

# Weyl Group Multiple Dirichlet Series and Gelfand-Tsetlin Patterns

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## 1 Introduction

An *L-function*, as the term is generally understood, is a Dirichlet series in one complex variable  $s$  with an Euler product that has (at least conjecturally) analytic continuation to all complex  $s$  and a functional equation under a single reflection  $s \mapsto 1 - s$ . The coefficients are in particular multiplicative.

By contrast *Weyl group multiple Dirichlet series* are a new class of Dirichlet series that differ from L-functions in two ways. First, although the coefficients of the series are not multiplicative, they are almost multiplicative, with the multiplicativity twisted by  $n$ -th power residue symbols (see (3) below). Second, they are Dirichlet series in several complex variables  $s_1, \dots, s_r$  that have (at least conjecturally) meromorphic continuation to all  $\mathbb{C}^r$  and groups of functional equations that are finite reflection groups. It is expected that many such series can be identified with the Whittaker coefficients of Eisenstein series on metaplectic groups and in this light, the twisted multiplicativity is just a natural generalization of multiplicativity to the metaplectic group. However current methods of study make limited use of this connection, which is less directly useful than the connection between L-functions and nonmetaplectic Eisenstein series in the Langlands-Shahidi method. One reason is that the metaplectic Eisenstein series in question usually do not have unique Whittaker models.

Weyl group multiple Dirichlet series were introduced formally in Brubaker, Bump, Chinta, Friedberg and Hoffstein [4] but particular instances and related multiple Dirichlet series can be found in earlier papers such as Siegel [27], Goldfeld and Hoffstein [18], Kubota [21], Bump, Friedberg and Hoffstein [9], Chinta [11] and Peter [23], [24]. Essential data in describing such a series in  $r$  complex variables are a root system  $\Phi$  of rank  $r$  with Weyl group  $W$ , a fixed integer  $n > 1$ , and a global ground field

$F$  containing the  $n$ -th roots of unity; in some of the literature (including this paper) the ground field  $F$  is assumed to contain the  $2n$ -th roots of unity. In the authors' papers [5], [6], for  $n$  sufficiently large (the “stable case”) we exhibited a class of such series whose coefficients involve  $n$ -th order Gauss sums in a way that is governed by the geometry of  $\Phi$ , and showed that the series had meromorphic continuation to all of  $\mathbb{C}^r$  and a group of functional equations isomorphic to  $W$ .

If  $n$  is small, the situation is more complex. When  $\Phi = A_r$  we described a multiple Dirichlet series in [8] whose coefficients are sums of products of Gauss sums, with the individual terms indexed by Gelfand-Tsetlin patterns. When  $n = 1$ , or  $n = 2$  and  $r \leq 5$ , or  $r \leq 2$  and  $n$  is arbitrary, we proved the continuation and functional equations of these series, and when  $n = 1$  or  $r \leq 2$  we established the link mentioned above with Eisenstein series on the  $n$ -fold metaplectic cover of  $GL(r+1)$ . In Theorem 1 of this paper we will prove the analytic continuation and functional equations of these multiple Dirichlet series for all  $n$  and  $r$ , which were conjectured in [8], Conjecture 1. Even in the case  $r = 2$ , the functional equation given here is more general than that deduced from [8] and also sharper, being equivalent to a full description of the scattering matrix.

Although the multiple Dirichlet series becomes more complex when  $n$  is small, in the case  $n = 1$ , the  $n$ -th order Gauss sums collapse to Ramanujan sums whose evaluations are more easily understood. Indeed it was shown in [8] that using an identity of Tokuyama [29], the multiple Dirichlet series reduces to the formula of Shintani [26] and Casselman and Shalika [10] for the Whittaker coefficients of Eisenstein series. See [8], Section 2, for a more detailed discussion.

Beyond this relationship with the Shintani-Casselman-Shalika formula when  $n = 1$ , the case of arbitrary  $n$  treated here has striking new features, notably the appearance of  $n$ -th order Gauss sums that remove the immediate connection with the Weyl character formula. There is reason to hope that this connection can be restored if the L-group  ${}^L G$  is replaced by a quantum group. But even then, formulation of the Shintani-Casselman-Shalika formula as a statement in representation theory must be changed since the metaplectic Whittaker models are not unique.

Our main result, Theorem 1 below, is global. Purely local statements (that is, for a fixed prime) that imply this result were given by the authors in [7], where the reduction to the local case was carried out. Several more such local statements are presented in the next Section. But despite strong signs of connections with representation theory, these are statements of combinatorics, rather than representation theory, and the proof we give makes heavy use of combinatorial ideas along with some number theory but not of representation theory. Thus the usual ways of proving the Shintani-Casselman-Shalika formula give no help.

Despite this caveat, the definition of the multiple Dirichlet series coefficients provided in the following pages in terms of Gelfand-Tsetlin patterns still retains representation-theoretic significance. As pointed out to us by Chinta and Gunnells, the arrays of coefficients in (6) that we derive from the Gelfand-Tsetlin patterns agree with an array associated by Berenstein and Zelevinsky [1] and Littelmann [22] to a vertex in Kashiwara's crystal graph having highest weight associated to the top row of the Gelfand-Tsetlin pattern. This is the array that is denoted  $\Gamma$  or  $\Delta$  here and in [8], and in Littelmann it is (coincidentally) also denoted  $\Delta$ ; see Section 5 of [22].

In Littelmann's paper these patterns are defined for all simple complex Lie algebras, and also Kac-Moody Lie algebras of rank two. For finite-dimensional simple Lie algebras, Littelmann shows that once one has fixed a minimal decomposition of the long Weyl group element into simple reflections, one can read off such an array from the crystal graph of an associated highest weight representation, and he shows this statement has a natural modification for at least some affine Lie algebras. Preliminary investigation suggests that this point of view leads to a definition of Weyl group multiple Dirichlet series for other Weyl groups, possibly even infinite Kac-Moody Weyl groups. Moreover, we hope that this point of view will lead to a resuscitation of the Casselman-Shalika formula in the context of metaplectic groups.

The original motivation for introducing Weyl group multiple Dirichlet series came from applications to number theory (for a recent example, see Chinta [11]). The potential for further applications remains an important motivation, particularly if the theory can be extended to Kac-Moody root systems. But due to the unexpected connections with representation theory and combinatorics, it now seems important to develop the theory independently of these applications.

Two recent series of papers are devoted to the development of the theory of Weyl group multiple Dirichlet series. One series of foundational papers consists of [4], Brubaker and Bump [2], Brubaker, Bump and Friedberg [5], [6], [7], Brubaker, Bump, Friedberg and Hoffstein [8] and this paper. The other foundational series consists of Chinta and Gunnells [14], [15], [16] and Chinta, Friedberg and Gunnells [13]. It is clear that that these two sets of papers are describing the same families of Dirichlet series, but this is not yet proved, nor is the conjectured relationship with the theory of metaplectic Eisenstein series. The two series of foundational papers are based on different approaches, prove different results, and are thus complementary to each other.

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## 2 Definitions and Main Result

In this section, we recall the definition of the Dirichlet series in question, and state the theorem that we will prove. In Section 3, we will outline the proof by presenting a series of statements that imply Theorem 1; we hope to thereby make the proof accessible, and also to convey some of the richness of the combinatorial structures that we have observed.

Let  $F$  be an algebraic number field containing the group  $\mu_{2n}$  of  $2n$ -th roots of unity. Let  $S$  be a finite set of places of  $F$  containing the archimedean ones and all those ramified over  $\mathbb{Q}$  (in particular those dividing  $n$ ). We choose  $S$  large enough that the ring  $\mathfrak{o}_S$  of  $S$ -integers is a principal ideal domain. We embed  $F$  and  $\mathfrak{o}_S$  into  $F_S = \prod_{v \in S} F_v$ . The multiple Dirichlet series of type  $A_r$  defined in [8] has the form

$$Z_\Psi(\mathbf{s}; \mathbf{m}) = Z_\Psi(\mathbf{s}; \mathbf{m}; A_r) = Z_\Psi(s_1, \dots, s_r; m_1, \dots, m_r) = \sum_{0 \neq c_i \in \mathfrak{o}_S / \mathfrak{o}_S^\times} H\Psi(c_1, \dots, c_r; m_1, \dots, m_r) \mathbb{N}_{c_1}^{-2s_1} \dots \mathbb{N}_{c_r}^{-2s_r}, \quad (1)$$

where the sum is over nonzero elements of  $\mathfrak{o}_S$  modulo the action of units; since  $\mathfrak{o}_S$  is a principal ideal domain, the sum is essentially over ideals. Here  $H\Psi$  is the product of two functions  $H$  and  $\Psi$  of  $c_1, \dots, c_r$  and  $m_1, \dots, m_r$  (nonzero elements of  $\mathfrak{o}_S$ ), whose definitions we will recall below. We are denoting  $\mathbf{s} = (s_1, \dots, s_r)$  and  $\mathbf{m} = (m_1, \dots, m_r)$ . We will prove here Conjecture 1 of [8]; that is, we will prove

**Theorem 1** *The function  $Z_\Psi(\mathbf{s}; \mathbf{m})$  has meromorphic continuation to all  $\mathbb{C}^r$  and group of functional equations isomorphic to the Weyl group  $S_{r+1}$  of  $A_r$  with the polar divisor and precise functional equations as described in [6].*

To complete this statement, we will next describe  $\Psi$  and  $H$ . The  $n$ -th power residue symbol  $\left(\frac{c}{d}\right)$  is defined when  $c$  and  $d$  are coprime elements of  $\mathfrak{o}_S$  and  $\gcd(n, d) = 1$ . It depends only on  $c$  modulo  $d$ , and satisfies the reciprocity law

$$\left(\frac{c}{d}\right) = (d, c)_S \left(\frac{d}{c}\right),$$

where  $(d, c)_S \in \mu_n$  denotes the  $S$ -Hilbert symbol, defined for  $c, d \in F_S^\times$ . (The properties of the Hilbert and power residue symbols in our notation are set out in [5].)

The function  $\Psi : (F_S^\times)^r \longrightarrow \mathbb{C}$  is assumed to satisfy

$$\Psi(\varepsilon_1 c_1, \dots, \varepsilon_r c_r) = \prod_{i=1}^r (\varepsilon_i, c_i)_S \left\{ \prod_{i < j} (\varepsilon_i, c_j)_S^{-1} \right\} \Psi(c_1, \dots, c_r) \quad (2)$$

when  $\varepsilon_1, \dots, \varepsilon_r \in \mathfrak{o}_S^\times (F^\times)^n$  and  $c_i \in F_S^\times$ . It is proved in [5] that the space of such functions is nonzero but finite-dimensional. Under the functional equations,  $\Psi$  will be affected by a scattering matrix, so the function  $Z_\Psi$  is essentially vector-valued.

The function  $H$ , unlike  $\Psi$ , is a fixed and definite one, to be described in terms of Gelfand-Tsetlin patterns as in [8]. In addition, a second multiple Dirichlet series described in terms of Gelfand-Tsetlin patterns – a sort of mirror image of the first – was introduced in [7], and it was shown there that the equality of the two series (essentially a local question) was sufficient to prove the functional equations. We will recapitulate both descriptions of coefficients below. Both coefficients  $H$  satisfy the multiplicativity

$$\frac{H(c_1 c'_1, \dots, c_r c'_r; m_1, \dots, m_r)}{H(c_1, \dots, c_r; m_1, \dots, m_r) H(c'_1, \dots, c'_r; m_1, \dots, m_r)} = \prod_{i=1}^r \left( \frac{c_i}{c'_i} \right) \left( \frac{c'_i}{c_i} \right) \prod_{i < j} \left( \frac{c_i}{c'_j} \right)^{-1} \left( \frac{c'_i}{c_j} \right)^{-1}. \quad (3)$$

when  $\gcd(c_1 \cdots c_r, c'_1 \cdots c'_r) = 1$ . Moreover if  $\gcd(m'_1 \cdots m'_r, c_1 \cdots c_r) = 1$  we will have the multiplicativity

$$\frac{H(c_1, \dots, c_r; m_1 m'_1, \dots, m_r m'_r)}{\left( \frac{m'_1}{c_1} \right)^{-1} \cdots \left( \frac{m'_r}{c_r} \right)^{-1} H(c_1, \dots, c_r; m_1, \dots, m_r)} = \quad (4)$$

This is a special case of the multiplicativity set out in [6] for general root systems. Due to this multiplicativity of  $H$ , specification of the coefficients is reduced to the specification of its  $p$ -part, where  $p$  is a prime of  $\mathfrak{o}_S$ . Thus to specify (either)  $H$  we are reduced to defining  $H(p^{k_1}, \dots, p^{k_r}; p^{l_1}, \dots, p^{l_r})$ . We turn to this next.

By a *Gelfand-Tsetlin pattern of rank  $r$*  we mean an array of integers

$$\mathfrak{T} = \left\{ \begin{array}{cccccc} a_{00} & a_{01} & a_{02} & \cdots & a_{0r} \\ & a_{11} & a_{12} & & a_{1r} \\ & & \ddots & & \ddots \\ & & & & a_{rr} \end{array} \right\} \quad (5)$$

where the rows interleave; that is,  $a_{i-1, j-1} \geq a_{i, j} \geq a_{i-1, j}$ . We will say that the pattern is *strict* if each row is strictly decreasing. We will associate with  $\mathfrak{T}$  two arrays  $\Gamma(\mathfrak{T})$  and  $\Delta(\mathfrak{T})$  whose entries may be *decorated* by boxing or circling certain of them. The entries in these arrays are

$$\gamma_{i, j} = \gamma_{i, j}(\mathfrak{T}) = \sum_{k=j}^r (a_{i, k} - a_{i-1, k}), \quad \delta_{i, j} = \delta_{i, j}(\mathfrak{T}) = \sum_{k=i}^j (a_{i-1, k-1} - a_{i, k}), \quad (6)$$

with  $1 \leq i \leq j \leq r$ . Thus in  $\Gamma$  we use a “right-hand rule” and accumulate row difference of  $\mathfrak{T}$  from right-to-left, as in (6). We also use a “right-hand rule” to describe the decoration: if  $a_{i,j} = a_{i-1,j-1}$  then we say  $\Gamma_{i,j}$  is *boxed*, and indicate this when we write the array by putting a box around it, while if  $a_{i,j} = a_{i-1,j}$  we say it is *circled* (and we circle it). But in  $\Delta$  we use a “left-hand rule” and accumulate row differences from left-to-right, also as in (6), and we use a left-hand rule to describe the decoration. This is the mirror image of the right-hand rule: if  $a_{i,j} = a_{i-1,j-1}$  then  $\Delta_{i,j}$  is circled, while if  $a_{i,j} = a_{i-1,j}$  it is boxed..

If  $m, c \in \mathfrak{o}_S$  with  $c \neq 0$  let

$$g(m, c) = \sum_{a \bmod c} \left(\frac{a}{c}\right) \psi\left(\frac{am}{c}\right),$$

where  $\psi$  is a character of  $F_S$  that is trivial on  $\mathfrak{o}_S$  and no larger fractional ideal. With  $p$  now fixed, for brevity let

$$g(a) = g(p^{a-1}, p^a), \quad h(a) = g(p^a, p^a),$$

and let  $q$  be the cardinality of  $\mathfrak{o}/p\mathfrak{o}$ . Let

$$G_\Gamma(\mathfrak{T}) = \prod_{1 \leq i \leq j \leq r} \begin{cases} g(\gamma_{ij}) & \text{if } \gamma_{ij} \text{ is boxed but not circled in } \Gamma(\mathfrak{T}); \\ q^{\gamma_{ij}} & \text{if } \gamma_{ij} \text{ is circled but not boxed;} \\ h(\gamma_{ij}) & \text{if } \gamma_{ij} \text{ neither circled nor boxed;} \\ 0 & \text{if } \gamma_{ij} \text{ both circled and boxed.} \end{cases}$$

It is clear from the definitions that  $\mathfrak{T}$  is nonstrict if and only if  $\Gamma(\mathfrak{T})$  has an entry that is both boxed and circled, so  $G_\Gamma(\mathfrak{T}) = 0$  for nonstrict patterns. Also let

$$G_\Delta(\mathfrak{T}) = \prod_{1 \leq i \leq j \leq r} \begin{cases} g(\delta_{ij}) & \text{if } \delta_{ij} \text{ is boxed but not circled in } \Delta(\mathfrak{T}); \\ q^{\delta_{ij}} & \text{if } \delta_{ij} \text{ is circled but not boxed;} \\ h(\delta_{ij}) & \text{if } \delta_{ij} \text{ neither circled nor boxed;} \\ 0 & \text{if } \delta_{ij} \text{ both circled and boxed.} \end{cases}$$

Let

$$k_\Gamma(\mathfrak{T}) = (k_1^\Gamma, \dots, k_r^\Gamma), \quad k_\Delta(\mathfrak{T}) = (k_1^\Delta, \dots, k_r^\Delta),$$

with

$$k_i^\Gamma = k_i^\Gamma(\mathfrak{T}) = \sum_{j=i}^r (a_{i,j} - a_{0,j})$$

and

$$k_i^\Delta = k_i^\Delta(\mathfrak{T}) = \sum_{j=r+1-i}^r (a_{0,j-r+1+i} - a_{r+1-i,j}).$$

Finally, let

$$H_{\Gamma}(p^{k_1}, \dots, p^{k_r}; p^{l_1}, \dots, p^{l_r}) = \sum_{k_{\Gamma}(\mathfrak{T})=(k_1, \dots, k_r)} G_{\Gamma}(\mathfrak{T})$$

and similarly

$$H_{\Delta}(p^{k_1}, \dots, p^{k_r}; p^{l_1}, \dots, p^{l_r}) = \sum_{k_{\Delta}(\mathfrak{T})=(k_1, \dots, k_r)} G_{\Delta}(\mathfrak{T}).$$

In these sums we need only sum over strict Gelfand-Tsetlin patterns since the terms corresponding to nonstrict patterns are zero. (However nonstrict patterns will be forced on us momentarily by the Schützenberger involution, which does not preserve strictness.) The definition of the multiple Dirichlet series  $Z_{\Psi}(\mathbf{s}; \mathbf{m})$  appearing in Theorem 1 is completed by taking  $H = H_{\Gamma}$ . The second multiple Dirichlet series mentioned above, introduced in [7], is obtained by taking  $H = H_{\Delta}$ .

### 3 Outline of the Proof

The proof of Theorem 1 involves many remarkable phenomena, and we wish to explain its structure in this section. To this end, will give the first of a succession of statements, each of which implies Theorem 1. Passing from each statement to the next is a nontrivial reduction that changes the nature of the problem to be solved. We will outline the ideas of these reductions here and tackle them in detail in subsequent sections.

**Statement A.** *We have  $H_{\Gamma} = H_{\Delta}$ .*

The proof that this implies Conjecture 1 of [8] is Theorem 1 of [7]. We review the idea of the proof. To prove the functional equations that  $Z(\mathbf{s}; \mathbf{m})$  is to satisfy, using the method described in [9], [4], [5] and [6] based on Bochner's convexity principle, one must prove meromorphic continuation to a larger region and a functional equation for each generator  $\sigma_1, \dots, \sigma_r$  of the Weyl group – the simple reflections. These act on the coordinates by

$$\sigma_i(s_j) = \begin{cases} 1 - s_j & \text{if } j = i, \\ s_i + s_j - \frac{1}{2} & \text{if } j = i \pm 1, \\ s_j & \text{if } |j - i| > 1. \end{cases}$$

We proceed inductively. Taking  $H = H_{\Gamma}$  as the definition of the series, all but one of these functional equations may be obtained by collecting the terms to produce

a series whose terms are multiple Dirichlet series of lower rank. To see this reduction, note that we have described the  $p$ -part of  $H$  as a sum over Gelfand-Tsetlin patterns, extended this to a definition to  $H(c_1, \dots, c_r; m_1, \dots, m_r)$  by (twisted) multiplicativity. Equivalently, one may define  $H(c_1, \dots, c_r; m_1, \dots, m_r)$  by specifying a Gelfand-Tsetlin pattern  $\mathfrak{T}_p$  for each prime; for all but finitely many  $p$  the pattern must be the minimal one

$$\left\{ \begin{array}{cccccc} r & & r-1 & & \cdots & 0 \\ & r-1 & & \cdots & & 0 \\ & & \ddots & & \ddots & \\ & & & 0 & & \end{array} \right\}.$$

Summing over such data with (for each  $p$ ) fixed top row (determined by the  $\text{ord}_p(m_i)$ ) and fixed row sums (determined by  $\text{ord}_p(c_i)$ ) gives  $H(c_1, \dots, c_r; m_1, \dots, m_r)$ . More precisely we may group the terms as follows. For each prime  $p$  of  $\mathfrak{o}_S$ , fix a partition  $\lambda_p$  of length  $r$  into unequal parts such that for almost all  $p$  we have  $\lambda_p = (r, r-1, \dots, 0)$ ; then collect the terms in which for each  $p$  the top row of  $\mathfrak{T}_p$  is  $\lambda_p$ . These produce an exponential factor times a term  $Z(\mathbf{s}; \mathbf{m}'; A_{r-1})$  where  $\mathbf{m}'$  depends on  $\lambda_p$  (for each  $p$ ). This expansion gives, by induction, the functional equations for the subgroup of  $W$  generated by  $\sigma_2, \dots, \sigma_r$ . Similarly starting with  $H = H_\Delta$  gives functional equations for the subgroup generated by  $\sigma_1, \dots, \sigma_{r-1}$ . Notice that these two sets of reflections generate all of  $W$ . If Statement A holds, then combining these analytic continuations and functional equations and invoking Bochner's convexity principle gives the required analytic continuation and functional equations. We refer to [7] for further details.

Since  $H_\Gamma$  and  $H_\Delta$  satisfy the same twisted multiplicativity, it suffices to work at powers of a single prime  $p$ . We see that there are two ways in which these coefficients differ. First, given a lattice point  $\mathbf{k} = (k_1, \dots, k_r)$  in the polytope defined by the Gelfand-Tsetlin patterns of given top row, there are two ways of attaching a set of Gelfand-Tsetlin patterns to  $\mathbf{k}$ :  $k_\Gamma(\mathfrak{T}) = \mathbf{k}$ , resp.  $k_\Delta(\mathfrak{T}) = \mathbf{k}$ . Second, given a pattern  $\mathfrak{T}$ , there are two ways of attaching numbers to it:  $G_\Gamma(\mathfrak{T})$ , resp.  $G_\Delta(\mathfrak{T})$ .

An attack on Statement A can be formulated using the *Schützenberger involution* on Gelfand-Tsetlin patterns. This involution was first defined for semi-standard Young tableaux in Schützenberger [25], then translated into the context of Gelfand-Tsetlin patterns in Kirillov and Berenstein [20]. It interchanges the functions  $k_\Delta$  and  $k_\Gamma$ , and one may thus formulate Statement A as saying that

$$\sum_{k_\Gamma(\mathfrak{T})=(k_1, \dots, k_r)} G_\Gamma(\mathfrak{T}) = \sum_{k_\Gamma(\mathfrak{T})=(k_1, \dots, k_r)} G_\Delta(\text{Sch}(\mathfrak{T})), \quad (7)$$

where  $\text{Sch}(\mathfrak{T})$  denotes the Schützenberger involution of  $\mathfrak{T}$ .

In many cases, for example if  $\mathfrak{T}$  is on the interior of the polytope of Gelfand-Tsetlin patterns with fixed  $k_\Gamma(\mathfrak{T})$ , it can be proved that  $G_\Gamma(\mathfrak{T}) = G_\Delta(\text{Sch}(\mathfrak{T}))$ . If this were always true there would be no need to sum in (7). In general, however, this is false. What *is* ultimately true is that the patterns may be partitioned into fairly small “packets” such that if one sums over a packet,  $\sum G_\Gamma(\mathfrak{T}) = \sum G_\Delta(\text{Sch}(\mathfrak{T}))$ . The packets, we observe, can be identified empirically in any given case, but are difficult to characterize in general, and not even uniquely determined in some cases. See [7].

To proceed further, we introduce the notion of a *short Gelfand-Tsetlin pattern*. By this we mean an array with just three rows

$$\mathfrak{t} = \left\{ \begin{array}{cccccc} l_0 & l_1 & l_2 & \cdots & l_{d+1} \\ & a_0 & a_1 & & a_d \\ & & b_0 & \cdots & b_{d-1} \end{array} \right\}. \quad (8)$$

where the rows are nonincreasing sequences of integers that interleave, that is,

$$l_i \geq a_i \geq l_{i+1}, \quad a_i \geq b_i \geq a_{i+1}. \quad (9)$$

We will refer to  $l_0, \dots, l_{d+1}$  as the *top* or *zero-th* row of  $\mathfrak{t}$ ,  $a_0, \dots, a_d$  as the *first* or *middle* row and  $b_0, \dots, b_{d-1}$  as the *second* or *bottom* row. We may assume that the top and bottom rows are strict, but we need to allow the first row to be nonstrict. We define the *weight*  $k$  of  $\mathfrak{t}$  to be the sum of the  $a_i$ .

If  $\mathfrak{t}$  is a short pattern we define another short pattern

$$\mathfrak{t}' = \left\{ \begin{array}{cccccc} l_0 & l_1 & l_2 & \cdots & l_{d+1} \\ & a'_0 & a'_1 & & a'_d \\ & & b_0 & \cdots & b_{d-1} \end{array} \right\}, \quad (10)$$

where

$$a'_j = \min(l_j, b_{j-1}) + \max(l_{j+1}, b_j) - a_j, \quad 0 < j < d, \quad (11)$$

$$a'_0 = l_0 + \max(l_1, b_0) - a_0, \quad a'_d = \min(l_d, b_{d-1}) + l_{d+1} - a_d. \quad (12)$$

We call  $\mathfrak{t}'$  the (Schützenberger) *involute* of  $\mathfrak{t}$ . To see why this definition is reasonable, note that if the top and bottom rows of  $\mathfrak{t}$  are specified, then  $a_i$  are constrained by the inequalities

$$\min(l_j, b_{j-1}) \geq a_j \geq \max(l_{j+1}, b_j), \quad 0 < j < d, \quad (13)$$

$$l_0 \geq a_0 \geq \max(l_1, b_0), \quad \min(l_d, b_{d-1}) \geq a_d \geq l_{d+1}. \quad (14)$$

These inequalities express the assumption that the three rows of the short pattern interleave. The array  $\mathfrak{t}'$  is obtained by reflecting  $a_j$  in its permitted range.

The Schützenberger involution of full Gelfand-Tsetlin patterns is built up from operations involving three rows at a time, based on the operation  $\mathfrak{t} \mapsto \mathfrak{t}'$  of short Gelfand-Tsetlin patterns. This is done  $\frac{1}{2}r(r+1)$  times to obtain the Schützenberger involution. Using this decomposition and induction, we prove that to establish Statement A one needs only the equivalence of two sums of Gauss sums corresponding to Gelfand-Tsetlin patterns that differ by a single involution. This allows us to restrict our attention, within Gelfand-Tsetlin patterns, to short patterns. To be more precise and to explain what must be proved, we make the following definitions.

By a short pattern *prototype*  $\mathfrak{S}$  of length  $d$  we mean a triple  $(\mathbf{l}, \mathbf{b}, k)$  specifying the following data: a top row consisting of an integer sequence  $\mathbf{l} = (l_0, \dots, l_{d+1})$ , a bottom row consisting of a sequence  $\mathbf{b} = (b_0, \dots, b_d)$ , and a positive integer  $k$ . It is assumed that  $l_0 > l_1 > \dots > l_{d+1}$ , that  $b_0 > b_1 > \dots > b_{d-1}$  and that  $l_i > b_i > l_{i+2}$ .

We say that a short pattern  $\mathfrak{t}$  of length  $d$  *belongs to the prototype*  $\mathfrak{S}$  if it has the prescribed top and bottom rows, and its weight is  $k$  (so  $\sum_i a_i = k$ ). By abuse of notation, we will use the notation  $\mathfrak{t} \in \mathfrak{S}$  to mean that  $\mathfrak{t}$  belongs to the prototype  $\mathfrak{S}$ . Prototypes were called *types* in [7], but we will reserve that term for a more restricted equivalence class of short patterns.

Given a short Gelfand-Tsetlin pattern, we may define two two-rowed arrays  $\Gamma_{\mathfrak{t}}$  and  $\Delta_{\mathfrak{t}}$ , which display information used in the evaluations we must make. These are defined analogously to the patterns associated with a full Gelfand-Tsetlin pattern, denoted  $\Gamma(\mathfrak{t})$  and  $\Delta(\mathfrak{t})$ , except that for  $\Gamma_{\mathfrak{t}}$  we use the right-hand rule on the first row, and the left-hand rule on the second row, and for  $\Delta_{\mathfrak{t}}$  we reverse these; in the full-pattern  $\Gamma(\mathfrak{t})$  we used the right-hand rule for every row, and in  $\Delta(\mathfrak{t})$  we used the left-hand rule for every row. Specifically

$$\Gamma_{\mathfrak{t}} = \left\{ \begin{array}{cccccc} \mu_0 & \mu_1 & \cdots & \mu_d \\ & \nu_0 & \cdots & \nu_{d-1} \end{array} \right\},$$

and

$$\Delta_{\mathfrak{t}} = \left\{ \begin{array}{cccccc} \kappa_0 & \kappa_1 & \cdots & \kappa_d \\ & \lambda_0 & \cdots & \lambda_{d-1} \end{array} \right\},$$

where

$$\mu_j = \sum_{k=j}^d (a_k - l_{k+1}), \quad \nu_j = \sum_{k=0}^j (a_k - b_k),$$

and

$$\kappa_j = \sum_{k=0}^j (l_k - a_k), \quad \lambda_j = \sum_{k=j}^{d-1} (b_k - a_{k+1}).$$

We also use the right-hand rule to describe the circling and boxing of the elements of the first row of  $\Gamma_{\mathbf{t}}$ , and the left-hand rule to describe the circling and boxing of elements of the bottom row, reversing these for  $\Delta_{\mathbf{t}}$ . This means we circle  $\mu_j$  if  $a_j = l_{j+1}$  and box  $\mu_i$  if  $a_j = l_j$ ; we circle  $\nu_j$  if  $b_j = a_j$  and box  $\nu_j$  if  $b_j = a_{j+1}$ . The boxing and circling rules are reversed for  $\Delta_{\mathbf{t}}$ : we box  $\kappa_j$  if  $\alpha_j = l_{j+1}$  and circle  $\alpha_i$  if  $\alpha_j = l_j$ ; we box  $\lambda_j$  if  $b_j = a_j$  and box  $\lambda_j$  if  $b_j = a_{j+1}$ .

We give an example to illustrate these definitions. Suppose that

$$\mathbf{t} = \left\{ \begin{array}{cccccc} 23 & 15 & 12 & 5 & 2 & 0 \\ & 20 & 12 & 5 & 4 & 2 \\ & & 14 & 9 & 5 & 3 \end{array} \right\}.$$

Then

$$\Gamma_{\mathbf{t}} = \left\{ \begin{array}{cccccc} 9 & \textcircled{4} & \textcircled{4} & 4 & \boxed{2} \\ & 6 & 9 & \textcircled{9} & 10 & \end{array} \right\}.$$

We have indicated the circling and boxing of entries. Now applying the involution,

$$\mathbf{t}' = \left\{ \begin{array}{cccccc} 23 & 15 & 12 & 5 & 2 & 0 \\ & 18 & 14 & 9 & 4 & 0 \\ & & 14 & 9 & 5 & 3 \end{array} \right\},$$

and

$$\Delta_{\mathbf{t}'} = \left\{ \begin{array}{cccccc} 5 & 6 & 9 & 10 & \boxed{12} \\ & \textcircled{4} & \textcircled{4} & 4 & 3 & \end{array} \right\}.$$

We observe the following points.

- The first row of  $\Gamma_{\mathbf{t}}$  is decreasing and the bottom row is increasing; these are reversed for  $\Delta_{\mathbf{t}'}$ , just as the boxing and circling conventions are reversed.
- The involution does not preserve strictness. If  $\mathbf{t}$  is strict, no element can be both boxed and circled, but if  $\mathbf{t}$  is not strict, an entry in the bottom row is both boxed and circled, and the same is true for  $\Delta_{\mathbf{t}'}$ : if  $\mathbf{t}'$  is not strict, then an entry in the bottom row of  $\Delta_{\mathbf{t}'}$  is both boxed and circled.

Let us define

$$G_{\Gamma}(\mathbf{t}) = \prod_{x \in \Gamma_{\mathbf{t}}} \begin{cases} g(x) & \text{if } x \text{ is boxed in } \Gamma_{\mathbf{t}}, \text{ but not circled;} \\ q^x & \text{if } x \text{ is circled, but not boxed;} \\ h(x) & \text{if } x \text{ is neither boxed nor circled;} \\ 0 & \text{if } x \text{ is both boxed and circled.} \end{cases}$$

Thus if  $\mathbf{t}$  is not strict, then  $G_{\Gamma}(\mathbf{t}) = 0$ . Similarly, let

$$G_{\Delta}(\mathbf{t}') = \prod_{x \in \Delta_{t'}} \begin{cases} g(x) & \text{if } x \text{ is boxed in } \Delta_{t'}, \text{ but not circled;} \\ q^x & \text{if } x \text{ is circled, but not boxed;} \\ h(x) & \text{if } x \text{ is neither boxed nor circled;} \\ 0 & \text{if } x \text{ is both boxed and circled.} \end{cases}$$

Thus in our examples,

$$G_{\Gamma}(\mathbf{t}) = h(9) \cdot q^4 \cdot q^4 \cdot h(4) \cdot g(2) \cdot h(6) \cdot h(9) \cdot q^9 \cdot h(10)$$

and

$$G_{\Delta}(\mathbf{t}') = h(5) \cdot h(6) \cdot h(9) \cdot h(12) \cdot g(12) \cdot q^4 \cdot q^4 \cdot h(4) \cdot h(3).$$

**Statement B.** *Let  $\mathfrak{S}$  be a short pattern prototype. Then*

$$\sum_{\mathbf{t} \in \mathfrak{S}} G_{\Gamma}(\mathbf{t}) = \sum_{\mathbf{t} \in \mathfrak{S}} G_{\Delta}(\mathbf{t}'). \quad (15)$$

The implication Statement B implies Statement A is established in [7]. The key idea is that, as noted above, the Schützenberger involution  $\text{Sch}(\mathfrak{T})$  is defined by repeatedly performing the involution on short patterns taking three rows at a time from  $\mathfrak{T}$ . Using this and induction, we show in [7] that Statement B implies (7), and Statement A follows. There is some bookkeeping involved and this is done carefully there.

To study Statement B, we first show in Theorems 2 and 3 below that for many  $\mathbf{t}$  (in some sense most), one actually has the equality

$$G_{\Gamma}(\mathbf{t}) = G_{\Delta}(\mathbf{t}'). \quad (16)$$

The proof requires some subtle combinatorics. Moreover (16) is not always true, so the summation in (15) is needed. The identity (15) seems deep – its proof uses properties of Gauss sums in the context of difficult combinatorial reasoning. Although (15) is a uniform statement valid for all  $n$ , the matching for one value of  $n$  may hold for a different number-theoretic reason than for another value of  $n$ .

Next we isolate a key case in which the summation in (15) is needed, and show that the general equality Statement B follows from handling this key case. We call the short pattern *totally resonant* if the bottom row repeats the top row, that is, if it has the form

$$\mathbf{t} = \left\{ \begin{array}{cccccc} l_0 & l_1 & l_2 & \cdots & l_{d+1} \\ & a_0 & a_1 & & a_d \\ & & l_1 & \cdots & l_d \end{array} \right\}. \quad (17)$$

Notice that this is a property of the prototype, whose data consists of the top and bottom rows, and the middle row sum.

**Statement C.** *Statement B is true for totally resonant short pattern prototypes.*

The fact that Statement C implies Statement B will be proved in this paper. To do so, given a short pattern (8), we will associate with its prototype a certain graph called the *cartoon* by connecting  $a_i$  to either  $l_i$  if  $i = 0$ , or else to either  $l_i$  or  $b_{i-1}$ , whichever is numerically closer to  $a_i$ . If  $b_{i-1} = l_i$ , then we connect  $a_i$  to both. We will also connect  $a_i$  to either  $l_{i+1}$  or  $b_i$ , whichever is numerically closer to  $a_i$ , or to both if they are equal, provided  $i < d$ ; and  $a_d$  is connected to  $l_{d+1}$ . The connected components of the cartoon are called *episodes*.

We will subdivide the prototype into smaller equivalence classes called *types*. Two patterns  $\mathbf{t}_1$  and  $\mathbf{t}_2$  have the same type if, first of all, they have the same prototype (hence the same cartoon), and if for each episode  $\mathcal{E}$ , the sum of the  $a_i$  that lie in  $\mathcal{E}$  are the same for  $\mathbf{t}_1$  and  $\mathbf{t}_2$ . Statement B follows from the stronger statement that (15) holds when we sum over a type. We will reduce this statement to a series of separate problems, one for each episode. But each of these problems will be reduced to a single common problem (Statement D below) that is equivalent to Statement B for totally resonant prototypes.

The reduction to the totally resonant case involves some quite fascinating combinatorial phenomena. A key point is a combinatorial “Snake Lemma,” which says the elements in  $\Gamma_{\mathbf{t}}$  can be matched up with the elements in  $\Delta_{\mathbf{t}}$  in a bijection that has quite surprising properties. (The “snakes” appear in graphing this bijection.)

In addition to such combinatorial phenomena, number theory enters as well, in particular in the “Knowability Lemma” (Proposition 10), which we now briefly discuss. The term “knowability” refers to the fact that Gauss sums such as  $g(a)$  when  $n \nmid a$  have known absolute values, but their arguments as complex numbers are still mysterious. They are “unknowable.” We will refer to an expression that is a product of terms of the form  $q^a$ ,  $h(a)$  and  $g(a)$  as “knowable” if it can given a closed expression as a polynomial in  $q$ . Thus  $g(a)$ , taken in isolation, is unknowable unless  $n|a$ , in which case  $g(a) = -q^{a-1}$ . But even if  $n \nmid a$  the product  $g(a)g(b)$  is knowable if  $n|a + b$  since then  $g(a)g(b) = q^{a+b-1}$ .

There is a strong tendency for the Gauss sums that appear in the terms  $G_{\Gamma}(\mathbf{t})$  (for short patterns  $\mathbf{t}$ ) or in  $G_{\Gamma}(\mathfrak{T})$  (for full Gelfand-Tsetlin patterns  $\mathfrak{T}$ ), to appear in knowable combinations. The Knowability Lemma give an explanation for this. Moreover it gives key information that is needed for the sequel. Stable patterns are an important exception. We recall that a pattern is *stable* if every entry (except those in the top row) equals one of the two directly above it. The stable patterns are in a sense the most important ones, since they are the *only* patterns that contribute in the stable case (when  $n$  is large). If the Gelfand-Tsetlin patterns with fixed top row are embedded into a Euclidean space, the stable patterns are the extremal

ones. The Gauss sums that appear in the stable terms are unknowable. Thus when  $r = 1$ ,  $Z(\mathbf{s}; \mathbf{m})$  is Kubota's Dirichlet series [21], and its use by Heath-Brown and Patterson [19] to study the distribution of cubic Gauss sums exploited precisely the appearance of such unknowable terms.

In order to prove Statement C, we must work with the evaluations of  $G_\Gamma(\mathbf{t})$  and  $G_\Delta(\mathbf{t}')$ . To do so, it is convenient to describe these evaluations by introducing some new notation and terminology. A  $\Gamma$ -*accordion* (of length  $d$  and weight  $s$ ) is an array of nonnegative integers

$$\mathbf{a} = \left\{ \begin{array}{ccccccc} s & \mu_1 & \mu_2 & \cdots & \mu_d \\ \nu_1 & \nu_2 & \cdots & \nu_d \end{array} \right\}, \quad (18)$$

in which the first row is decreasing, the second increasing, and  $\mu_i + \nu_i = s$ . We will sometimes write

$$\mu_0 = \nu_{d+1} = s, \quad \nu_0 = \mu_{d+1} = 0. \quad (19)$$

Also, by a  $\Delta$ -*accordion* (of length  $d + 1$  and weight  $s$ ) we mean an array

$$\mathbf{a}' = \left\{ \begin{array}{cccccc} \nu_1 & \nu_2 & \cdots & \nu_d & s \\ \mu_1 & \mu_2 & \cdots & \mu_d \end{array} \right\}, \quad (20)$$

where the first row is increasing, the bottom decreasing, and  $\mu_i + \nu_i = s$ . We will make use of the map  $\mathbf{a} \mapsto \mathbf{a}'$  that takes  $\Gamma$ -accordions to  $\Delta$ -accordions.

The significance of these definitions is that if  $\mathbf{t}$  is a totally resonant short Gelfand-Tsetlin pattern, then its  $\Gamma$ -array is a  $\Gamma$ -accordion. Moreover the  $\Delta$ -array of  $\mathbf{t}'$  is the  $\Delta$ -accordion  $\mathbf{a}'$ .

To account for the difference in the contributions coming from circled or boxed entries in the arrays coming from short Gelfand-Tsetlin patterns, we will consider accordions that are *decorated* by circling or boxing certain elements. The circling and boxing in accordions follows certain rules because they arise from a totally resonant short pattern.

- No entry of the first row is both boxed and circled. An entry of the bottom row may be both boxed and circled, in which case we say the accordion is *nonstrict*.
- In a  $\Gamma$ -accordion, a bottom row entry is circled if and only if the entry above it and to the left is circled, and a bottom row entry is boxed if and only if the entry above it and to the right is boxed. Thus in (18),  $\nu_i$  is circled if and only if the  $\mu_{i-1}$  is circled, and  $\nu_i$  is boxed if and only if  $\mu_i$  is boxed.

- In a  $\Delta$ -accordion, a bottom row entry is circled if and only if the entry above it and to the right is circled, and a bottom row entry is boxed if and only if the entry above it and to the left is boxed.
- In either (18) or (20)  $\mu_i$  is circled if and only if  $\mu_i = \mu_{i+1}$ , and  $\nu_i$  is circled if and only if  $\nu_i = \nu_{i-1}$ . (But note that  $\mu_i$  is in the first row in (18) but in the second row in (20).) Invoking (19), special cases of this rule are that  $s$  is circled if and only if  $s = \mu_1$ ,  $\mu_d$  is circled if and only if  $\mu_d = 0$ , and  $\nu_1$  is circled if and only if  $\nu_1 = 0$ .
- There is no corresponding rule for the boxing. Thus the circling is determined by  $\mathbf{a}$  but the boxing is not.

We also note that the Knowability Lemma mentioned above allows us to assume that  $n|s$ .

Our goal is to systematically describe all decorated accordions that arise and their corresponding evaluations, so that we may sum over them and establish Statement C. If  $\mathbf{a}$  is a decorated accordion (of either type), then in view of the second and third rules, the decoration of the second row is determined by the decoration of the top row. We encode this by a *signature*, which is by definition a string  $\sigma = \sigma_0 \cdots \sigma_d$ , where each  $\sigma_i$  is one of the symbols  $\circ, \square$  or  $*$ . We associate a signature with a decorated accordion by taking  $\sigma_i = \circ$  if  $\mu_i$  is circled in the first row (with  $\mu_0 = s$ , of course),  $\square$  if  $\mu_i$  is boxed, and  $*$  if it is neither circled or boxed. We say the accordion  $\mathbf{a}$  and the signature  $\sigma$  are *compatible* if the following *circling compatibility condition* is satisfied (for conformity with the rules already stated for the decorations). For  $\Gamma$ -accordions labeled as in (18) the condition is

$$\sigma_i = \circ \text{ if and only if } \mu_i = \mu_{i+1}. \quad (21)$$

In view of (19), if  $i = 0$ , this means  $s = \mu_1$ , and if  $i = d$  it means  $\mu_d = 0$ . For  $\Delta$ -accordions labeled as in (20) the condition is

$$\sigma_i = \circ \text{ if and only if } \nu_i = \nu_{i-1},$$

which means that  $\sigma_0 = \circ$  if and only if  $\nu_1 = 0$ , and  $\sigma_d = \circ$  if and only if  $s = \nu_1$ .

Since the signature determines the decoration, we will denote by  $\mathbf{a}_\sigma$  the decorated accordion, where  $\sigma$  is a signature compatible with the accordion  $\mathbf{a}$ . We will apply the same signature  $\sigma$  to the involute  $\mathbf{a}'$ . Thus if  $\sigma = * \square * * \circ *$  and

$$\mathbf{a} = \left\{ \begin{array}{cccccc} 9 & 7 & 6 & 4 & 2 & 2 \\ & 2 & 3 & 5 & 7 & 7 \end{array} \right\}$$

then

$$\mathbf{a}_\sigma = \left\{ \begin{array}{cccccc} 9 & \boxed{7} & 6 & 5 & \textcircled{2} & 2 \\ \boxed{2} & & 3 & 5 & 7 & \textcircled{7} \end{array} \right\},$$

which is a decorated  $\Gamma$ -accordion, while

$$\mathbf{a}'_\sigma = \left\{ \begin{array}{cccccc} 2 & \boxed{3} & 5 & 7 & \textcircled{7} & 9 \\ 7 & & \boxed{6} & 4 & \textcircled{2} & 2 \end{array} \right\},$$

which is a decorated  $\Delta$ -accordion. Observe that the signature encodes the decoration of the first row, and since we use the same signature in both  $\mathbf{a}$  and  $\mathbf{a}'$ , it follows that the location of the boxes and circles in the first row is the same for  $\mathbf{a}$  as for the  $\Delta$ -accordion  $\mathbf{a}'$ . However the decoration of the bottom rows are different. This is because the boxing and circling rules are different for  $\Gamma$ -accordions and  $\Delta$ -accordions.

Now if  $\mathbf{a}_\sigma$  is a decorated  $\Gamma$ -accordion, let

$$\mathcal{G}_\Gamma(\mathbf{a}, \sigma) = \mathcal{G}_\Gamma(\mathbf{a}_\sigma) = \prod_{x \in \mathbf{a}} \begin{cases} g(x) & \text{if } x \text{ is boxed in } \mathbf{a}_\sigma \text{ (but not circled),} \\ q^x & \text{if } x \text{ is circled (but not boxed),} \\ h(x) & \text{if } x \text{ is neither boxed nor circled,} \\ 0 & \text{if } x \text{ is both boxed and circled.} \end{cases}$$

The notations  $\mathcal{G}_\Gamma(\mathbf{a}_\sigma)$  and  $\mathcal{G}_\Gamma(\mathbf{a}, \sigma)$  are synonyms; we will prefer the former when working with the free abelian group on the decorated accordions, the latter when  $\mathbf{a}$  is fixed and  $\sigma$  is allowed to vary.

If  $\mathbf{a}'_\sigma$  is a decorated  $\Delta$ -accordion, we define  $\mathcal{G}_\Delta(\mathbf{a}'_\sigma)$  by the same formula. We retain the subscripts  $\Gamma$  and  $\Delta$  since  $\mathcal{G}_\Gamma$  and  $\mathcal{G}_\Delta$  have different domains. Thus in the last example

$$\mathcal{G}_\Gamma(\mathbf{a}_\sigma) = h(9)g(7)g(2)h(6)h(3)h(5)h(5)q^2h(7)h(2)q^7,$$

$$\mathcal{G}_\Delta(\mathbf{a}'_\sigma) = h(2)h(7)g(3)g(6)h(5)h(4)h(7)q^2q^7h(2)h(9).$$

Now let positive integers  $s, c_0, c_1, \dots, c_d$  be given. By the  $\Gamma$ -*resotope* (of length  $d$ ), to be denoted  $\mathcal{A}_s^\Gamma(c_0, c_1, \dots, c_d)$ , we mean the sum, in the free abelian group  $\mathfrak{Z}_\Gamma$  on the set of decorated accordions, of such  $\mathbf{a}_\sigma$  such that the parameters in  $\mathbf{a}$  satisfy

$$0 \leq s - \mu_1 \leq c_0, \quad 0 \leq \mu_1 - \mu_2 \leq c_1, \quad \dots, \quad 0 \leq \mu_d \leq c_d, \quad (22)$$

and

$$\sigma_i = \begin{cases} \circ & \text{if } \mu_i - \mu_{i+1} = 0; \\ \square & \text{if } \mu_i - \mu_{i+1} = c_i; \\ * & \text{if } 0 < \mu_i - \mu_{i+1} < c_i. \end{cases}$$

If  $\mathcal{A} = \mathcal{A}_s^\Gamma(c_0, c_1, \dots, c_d)$ , by abuse of notation we may write  $\mathbf{a}_\sigma \in \mathcal{A}$  to mean that  $\mathbf{a}_\sigma$  appears with nonzero coefficient in  $\mathcal{A}$  as described above. Let  $\mathcal{A}'$  be the image of  $\mathcal{A}$  under the involution  $\mathbf{a}_\sigma \mapsto \mathbf{a}'_\sigma$ ; we call  $\mathcal{A}'$  a  $\Delta$ -resotope.

The set of  $\Gamma$ -accordions of the form (18), embedded into Euclidean space by mapping  $\mathbf{a} \mapsto (\mu_1, \dots, \mu_d)$ , may be regarded as the set of lattice points in a polytope. The resotopes we have just described correspond to these points with compatible decorations and signatures attached. We will sometimes discuss the geometry of the underlying polytope without making explicit note of the additional data attached to each point.

Given a totally resonant type, we will prove in the Corollary to Proposition 12 below that the array  $\Gamma_t$  runs through a resotope  $\mathcal{A}$ , and  $\Delta_t$  runs through  $\mathcal{A}'$ . This allows us to pass from Statement C to the following statement.

**Statement D.** *Let  $\mathcal{A}$  be a  $\Gamma$ -resotope. Then*

$$\sum_{\mathbf{a}_\sigma \in \mathcal{A}} \mathcal{G}_\Gamma(\mathbf{a}_\sigma) = \sum_{\mathbf{a}'_\sigma \in \mathcal{A}'} \mathcal{G}_\Delta(\mathbf{a}'_\sigma). \quad (23)$$

We turn to the proof of Statement D. We will show that if  $\mathbf{a}$  lies in the interior of the resotope, then its signature is just  $* \cdots *$ , and we have  $\mathcal{G}_\Gamma(\mathbf{a}_\sigma) = h(s) \prod_{i=1}^d h(\mu_i) h(\nu_i) = \mathcal{G}_\Delta(\mathbf{a}'_\sigma)$ . This may fail, however, when  $\mathbf{a}$  is on the boundary, so this is the remaining obstacle to proving Statement D. The approach suggested in [7] is to try to partition the boundary into small “packets” such that the sums over each packet are equal. In practice one can carry this out in any given case, but giving a coherent theory of packets along these lines seems unpromising. First, the resohedra themselves are geometrically complex. Second, even when the resotope is geometrically simple, the identification of the packets can be perplexing, and devoid of any apparent pattern. An example is done at the end of [7].

Geometrically, a resohedron is a figure obtained from a simplex by chopping off some of the corners; the pieces removed are themselves simplices. But the resulting polytopes are quite varied. Figure 1 shows a resohedron with five pentagonal faces and three triangular ones. To avoid these geometric difficulties we develop an approach, based on the Principle of Inclusion-Exclusion, that allows us to replace the complicated geometry of a general polytope with the simple geometry of a simplex.

The process of passing from the simplex of all accordions to an arbitrary resotope is complex. Indeed, as one chops corners off the simplex of all accordions to obtain a general resotope, interior accordions become boundary accordions, so their signatures change. Sometimes the removed simplices overlap, so one must restore any part that has been removed more than once. As we shall show, there is nonetheless a good way of handling it. Before we formulate this precisely, let us consider an example.

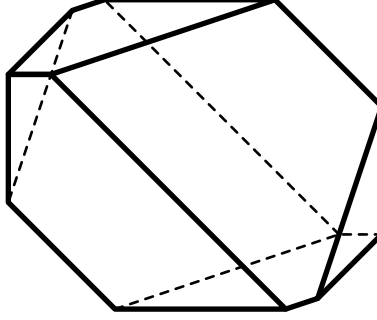


Figure 1: A resohedron, when  $d = 3$ .

The set of all  $\Gamma$ -resohedra, with  $d = 2$  and fixed value  $s$  is represented in Figure 2 by the triangle  $\Delta \mathbf{abc}$  with vertices

$$\mathbf{a} = \begin{Bmatrix} s & s & 0 \\ 0 & s & \end{Bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} s & 0 & 0 \\ s & s & \end{Bmatrix}, \quad \mathbf{c} = \begin{Bmatrix} s & s & s \\ 0 & 0 & \end{Bmatrix}.$$

We are concerned with the shaded resotope  $\mathcal{A} = \mathcal{A}_s^\Gamma(c_0, c_1, \infty)$ , which is obtained by truncating the simplex  $\Delta \mathbf{abc}$  by removing  $\Delta \mathbf{aeg}$  and  $\Delta \mathbf{dbh}$ . We use  $\infty$  to mean any value of  $c_2$  that is so large that the inequality  $\mu_2 \leq c_2$  is automatically true (and strict) for all  $\Gamma$ -accordions; indeed any  $c_2 > s$  can be replaced by  $\infty$  without changing  $\mathcal{A}$ .

Then, since the  $\Delta \mathbf{def}$  has been removed twice, it must be restored, and we may write

$$\mathcal{A} = \Delta \mathbf{abc} - \Delta \mathbf{aeg} - \Delta \mathbf{dbh} + \Delta \mathbf{def}.$$

Now in this equation,  $\Delta \mathbf{def}$  (for example) should be regarded as an element of  $\mathfrak{Z}_\Gamma$ , and in addition to specifying its *support* – its underlying set – we must also specify what signatures occur with each accordion that appears in it, and with what sign. For example, in  $\Delta \mathbf{def}$  the accordion  $\mathbf{f}$  will occur with four different signatures: the actual contribution of  $\mathbf{f}$  to  $\Delta \mathbf{def}$  is

$$\mathbf{f}_{\square\square*} - \mathbf{f}_{\square**} - \mathbf{f}_{*\square*} + \mathbf{f}_{***}.$$

Now let us give a formal description of this setup. A signature  $\eta$  is called *nodal* if each  $\eta_i$  is either  $\circ$  or  $\square$ . We fix a nodal signature  $\eta$ . Let  $\text{CP}_\eta(c_0, \dots, c_d)$  be the “cut and paste” simplex, which is the set of  $\Gamma$ -accordions

$$\mathbf{a} = \begin{Bmatrix} s & \mu_1 & \cdots & \mu_d \\ \nu_1 & \cdots & \nu_d & \end{Bmatrix} \quad (24)$$

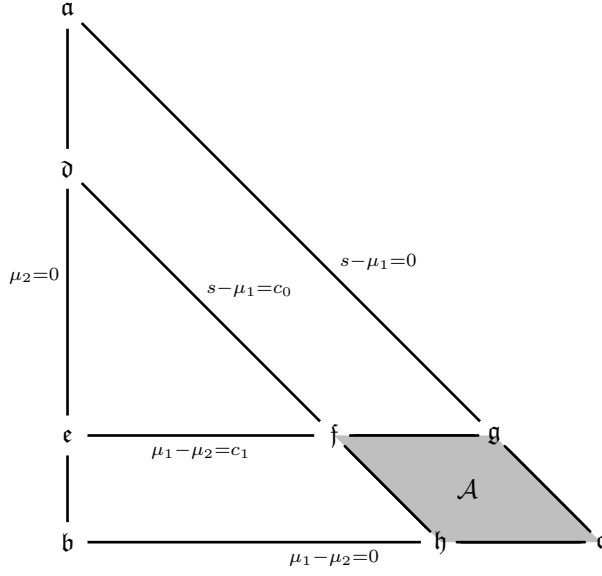


Figure 2: Inclusion-Exclusion.

that satisfy the inequalities

$$\mu_i - \mu_{i+1} \geq c'_i, \quad c'_i = \begin{cases} c_i & \text{if } \sigma_i = \square, \\ 0 & \text{if } \sigma_i = \circ, \end{cases} \quad (25)$$

To see that this is truly a simplex, embed it into Euclidean space via the map

$$\mathbf{a} \longmapsto (a_0, a_1, \dots, a_d), \quad a_i = \mu_i - \mu_{i+1} - c'_i.$$

The image of this map is the set of integer points in the simplex defined by the inequalities

$$a_i \geq 0, \quad a_0 + \dots + a_d \leq N, \quad N = s - \sum c'_i.$$

Thus in the above example, with  $\eta = \square\square\circ$  we have

$$\begin{aligned} \Delta \mathbf{abc} &= \text{CP}_{**\circ}(c_0, c_1, \infty), & \Delta \mathbf{aeg} &= \text{CP}_{*\square\circ}(c_0, c_1, \infty), \\ \Delta \mathbf{dbh} &= \text{CP}_{\square*\circ}(c_0, c_1, \infty), & \Delta \mathbf{def} &= \text{CP}_{\square\square\circ}(c_0, c_1, \infty). \end{aligned}$$

If  $\sigma$  and  $\tau$  are signatures, we say that  $\tau$  is a *subsignature* of  $\sigma$  if  $\tau_i = \sigma_i$  whenever  $\tau_i \neq *$ , and we write  $\tau \subset \sigma$  in this case. In other words,  $\tau \subset \sigma$  if  $\tau$  is obtained from  $\sigma$  by changing some  $\square$ 's or  $\circ$ 's to  $*$ 's. If  $\tau$  is a signature, we will denote  $\text{sgn}(\tau) = (-1)^\varepsilon$  where  $\varepsilon$  is the number of boxes in  $\tau$ .

Returning to the general case, let  $\mathbf{a}$  be a  $\Gamma$ -accordion, and let  $\sigma$  be a compatible signature for  $\mathbf{a}$ . Define

$$\Lambda_{\Gamma}(\mathbf{a}, \sigma) = \sum_{\mathbf{a}\text{-compatible } \tau \subset \sigma} \text{sgn}(\tau) \mathcal{G}_{\Gamma}(\mathbf{a}, \tau), \quad \Lambda_{\Delta}(\mathbf{a}', \sigma) = \sum_{\mathbf{a}\text{-compatible } \tau \subset \sigma} \text{sgn}(\tau) \mathcal{G}_{\Delta}(\mathbf{a}', \tau).$$

In the definition of subsignature we allow either  $\square$ 's or  $\circ$ 's to be changed to  $*$ 's (needed for later purposes). But in this summation, because  $\tau$  and  $\sigma$  are both required to be compatible with the same accordion  $\mathbf{a}$ , only  $\square$ 's are changed to  $*$  between  $\sigma$  and  $\tau$  in the  $\tau$  that appear in this definition.

Thus in the above example

$$\Lambda_{\Gamma}(\mathbf{f}, \square\square*) = \mathcal{G}_{\Gamma}(\mathbf{f}_{\square\square*}) - \mathcal{G}_{\Gamma}(\mathbf{f}_{\square**}) - \mathcal{G}_{\Gamma}(\mathbf{f}_{*\square*}) + \mathcal{G}_{\Gamma}(\mathbf{f}_{***}).$$

Now let  $\eta$  be a nodal signature, and we may take  $c_i = \infty$  if  $\eta_i = \circ$ . Let  $\mathbf{a} \in \text{CP}_{\eta}(c_0, \dots, c_d)$ . Let  $\sigma = \sigma(\mathbf{a})$  be the subsignature of  $\eta$  obtained by changing  $\eta_i$  to  $*$  when the inequality (25) is strict.

**Statement E.** *Assume that  $n|s$ . We have*

$$\sum_{\mathbf{a} \in \text{CP}_{\eta}(c_0, \dots, c_d)} \Lambda_{\Gamma}(\mathbf{a}, \sigma) = \sum_{\mathbf{a} \in \text{CP}_{\eta}(c_0, \dots, c_d)} \Lambda_{\Delta}(\mathbf{a}', \sigma). \quad (26)$$

We reiterate that in this sum  $\sigma$  depends on  $\mathbf{a}$ , and we have described the nature of the dependence above.

We have already noted that  $n|s$  can be imposed in Statement D and now we impose it explicitly. We will show in Section 10 that Statement E implies Statement D by application of the Inclusion-Exclusion principle. We hope for the purpose of this introduction, the above example will make that plausible. In that example, the four triangles  $\Delta \mathbf{abc}$ ,  $\Delta \mathbf{aeg}$ ,  $\Delta \mathbf{dbc}$  and  $\Delta \mathbf{def}$  are examples of cut and paste simplices.

We have already mentioned that in the context of Statements B, C or D, it is empirically possible to partition the sum into a disjoint union of smaller units called *packets* such that the identity is true when summation is restricted to a packet. Yet it is also true that in those contexts, a general rule describing the packets is notoriously slippery to nail down. However, in the context of Statement E we are able to describe the packets explicitly. The  $d$ -dimensional simplex  $\text{CP}_{\eta}(c_0, \dots, c_d)$  is partitioned into *facet*, which are subsimplices of lower dimension. Specifically, there are  $\binom{d+1}{f+1}$  facets that are simplices of dimension  $f$ ; we will call these *f-facets*. We will define the packets so that if  $\mathbf{a}$  lies on the interior of an  $f$ -facet, then the packet containing  $\mathbf{a}$  has  $\binom{d+1}{f+1}$  elements, one chosen from the interior of each  $r$ -facet.

Let us make this precise. First we observe the following description of the  $d + 1$  vertices  $\mathbf{a}_i$  of  $\text{CP}_\eta(c_0, \dots, c_d)$ . If  $0 \leq i \leq d$  let  $\mathbf{a}_i$  be the accordion whose coordinates  $\mu_i$  are determined by the equations

$$\mu_j - \mu_{j+1} = c'_j, \quad \text{for all } 0 \leq j \leq d \text{ with } j \neq i.$$

A *closed (resp. open)  $f$ -facet* of  $\text{CP}_\eta(c_0, \dots, c_d)$  will be the set of integer points in the closed (resp. open) convex hull of a subset with cardinality  $f + 1$  of the set  $\{\mathbf{a}_0, \dots, \mathbf{a}_d\}$  of vertices. Clearly every element of  $\text{CP}_\eta(c_0, \dots, c_d)$  lies in a unique open facet.

We associate the facets with subsignatures  $\sigma$  of  $\eta$ ; if  $\sigma$  is obtained by replacing  $\eta_i$  ( $i \in S$ ) by  $*$ , where  $S$  is some subset of  $\{0, 1, 2, \dots, d\}$ , then we will denote the set of integer points in the closed (resp. open) convex hull of  $\mathbf{a}_i$  ( $i \in S$ ) by  $\overline{\mathcal{S}_\sigma}$  (resp.  $\mathcal{S}_\sigma$ ). The facet  $\overline{\mathcal{S}_\sigma}$  is itself a simplex, of dimension  $f$ . Thus if  $\sigma$  is a signature with exactly  $f + 1$   $*$ 's, we call  $\sigma$  an  *$f$ -signature* or an  *$f$ -subsignature of  $\eta$* .

Now if  $\sigma$  and  $\tau$  are  $f$ -subsignatures of  $\eta$ , then we will define a bijection  $\phi_{\sigma, \tau} : \overline{\mathcal{S}_\sigma} \rightarrow \overline{\mathcal{S}_\tau}$ . It is the unique affine linear map that takes the vertices of  $\overline{\mathcal{S}_\sigma}$  to the vertices of  $\overline{\mathcal{S}_\tau}$  in order. This means that if

$$S = \{s_0, \dots, s_f\}, \quad 0 \leq s_0 < s_1 < \dots < s_f \leq d$$

is the set of  $i$  such that  $\sigma_i = *$ , and similarly if

$$T = \{t_0, \dots, t_f\}, \quad 0 \leq t_0 < t_1 < \dots < t_f \leq d$$

is the set of  $i$  such that  $\tau_i = *$ , then  $\phi_{\sigma, \tau}$  takes  $\mathbf{a}_{s_i}$  to  $\mathbf{a}_{t_i}$ , and this map on vertices is extended by affine linearity to a map on all of  $\overline{\mathcal{S}_\sigma}$ .

It is obvious from the definition that  $\phi_{\sigma, \sigma}$  is the identity map on  $\mathcal{S}_\sigma$  and that if  $\sigma, \tau, \theta$  are  $f$ -subsignatures of  $\eta$  then  $\phi_{\tau, \theta} \circ \phi_{\sigma, \tau} = \phi_{\sigma, \theta}$ . This means that we may define an equivalence relation on  $\text{CP}_\eta(c_0, \dots, c_d)$  as follows. Let  $\mathbf{a}, \mathbf{b} \in \text{CP}_\eta(c_0, \dots, c_d)$ . Let  $\mathcal{S}_\sigma$  and  $\mathcal{S}_\tau$  be the (unique) open facets such that  $\mathbf{a} \in \mathcal{S}_\sigma$  and  $\mathbf{b} \in \mathcal{S}_\tau$ . Then  $\mathbf{a}$  is equivalent to  $\mathbf{b}$  if and only if  $\phi_{\sigma, \tau}(\mathbf{a}) = \mathbf{b}$ . The equivalence classes are called *packets*. It is clear from the definitions that the number  $f + 1$  of  $*$ 's in  $\sigma$  is constant for  $\sigma$  that appear in a packet  $\Pi$ , and we will call  $\Pi$  a  *$f$ -packet*. Clearly every  $f$ -packet contains exactly one element from each  $f$ -simplex.

**Statement F.** *Assume that  $n|s$ . Let  $\Pi$  be a packet. Then*

$$\sum_{\mathbf{a} \in \Pi} \Lambda_\Gamma(\mathbf{a}, \sigma) = \sum_{\mathbf{a}' \in \Pi} \Lambda_\Delta(\mathbf{a}', \sigma). \quad (27)$$

As in Statement E,  $\sigma$  depends on  $\mathbf{a}$  in this sum, and from the definition of packets, no  $\sigma$  appears more than once; in fact, if  $\Pi$  is an  $f$ -packet, then every  $f$ -subsignature  $\sigma$  of  $\eta$  appears exactly once on each side of the equation.

It is obvious that Statement F implies Statement E. There is one further sufficient condition that we call **Statement G**, but a proper formulation requires more notation than we want to give at this point. We will therefore postpone Statement G to the end of Section 11, describing it here in informal terms.

Due to the knowability property of the products of Gauss sums that make up  $\mathcal{G}_\Gamma(\mathbf{a}, \sigma)$ , these can be evaluated explicitly when  $n|s$  (Proposition 17) and this leads to an evaluation of  $\Lambda_\Gamma(\mathbf{a}, \sigma)$  and a similar evaluation of  $\Lambda_\Delta(\mathbf{a}', \sigma)$  (Theorems 6 and 7). The evaluation is in terms of sums of powers of  $q$  and  $q - 1$  parametrized by rather complicated combinatorial data. However, these evaluations also depend on the divisibility properties of the  $\mu_i$  that appear in the top row of  $\mathbf{a}$  given by (24); more precisely, if  $\Sigma$  is a subset of  $\{1, 2, \dots, d\}$ , let  $\delta_n(\Sigma; \mathbf{a})$  be 1 if  $n|\mu_i$  for all  $i \in \Sigma$  and 0 otherwise. Then there is a sum over certain such subsets  $\Sigma$  of  $\mathbf{a}$  – only those  $i$  such that  $\sigma_i \neq \circ$  can appear – and the terms that appear are with a coefficient  $\delta_n(\Sigma; \mathbf{a})$ . We recall that each  $\sigma$  appears only once on each side of (27), and hence  $\mathbf{a}$  is really a function of  $\sigma$ . Thus we are reduced to proving Statement G, amounting to an identity (59) in which there is first a sum over all  $f$ -subsignatures  $\sigma$  of  $\eta$ , and then a sum over subsets  $\Sigma$  of  $\{1, 2, \dots, d\}$ .

The identity (59) seems at first perplexing since  $\delta_n(\Sigma; \mathbf{a})$  depends on  $\mathbf{a}$ . It is not sufficient to simply interchange the order of summation since then  $\delta_n(\Sigma; \mathbf{a})$  will not be constant on the inner sum over  $\mathbf{a}$  (or equivalently  $\sigma$ ). However we are able to identify an equivalence relation that we call *concurrency* on pairs  $(\sigma, \Sigma)$  such that  $\delta_n(\Sigma; \mathbf{a})$  is constant on concurrency classes (Proposition 18). We will then establish a result that implies that some groups of terms from the same side of (59) involve concurrent data (Proposition 19). These concurrent data are called  $\Gamma$ -packs for the left-hand side or  $\Delta$ -packs for the right-hand side. Then we will need a rather more subtle result (Proposition 20) giving a bijection between the  $\Gamma$ -packs and the  $\Delta$ -packs that also matches concurrent data. With these combinatorial preparations, we prove (59) and therefore Statement G. We note that the ultimate equality that is established is not simply a formal matching of contributions on the two sides; indeed a matching  $\Gamma$ -pack and  $\Delta$ -pack may have different cardinalities but nonetheless sum to the same quantity.

## 4 Cartoons

**Proposition 1** (i) If  $n \nmid a$  then  $h(a) = 0$ , while if  $n|a$  we have

$$h(a+b) = q^a h(b), \quad g(a+b) = q^a h(b).$$

(ii) If  $n|a$  then

$$h(a) = \phi(p^a) = q^{a-1}(q-1), \quad g(a) = -q^{a-1},$$

while if  $n \nmid a$  then  $h(a) = 0$  and  $|g(a)| = q^{a-\frac{1}{2}}$ . If  $n \nmid a, b$  but  $n|a+b$  then

$$g(a)g(b) = q^{a+b-1}.$$

**Proof** This is easily checked using standard properties of Gauss sums.  $\square$

For the reduction to totally resonant prototypes – that is, the fact that Statement C implies Statement B – only (i) is used. The properties in (ii) become important later.

To define the cartoon, we will take a slightly more formal approach to the short Gelfand-Tsetlin patterns. Let

$$\Theta = \{(i, j) \in \mathbb{Z} \times \mathbb{Z} \mid 0 \leq i \leq 2, 0 \leq j \leq d+1-i\}.$$

We call this set the *substrate*, and divide  $\Theta$  into three *rows*, which are

$$\begin{aligned} \Theta_0 &= \{(0, j) \in \Theta \mid 0 \leq j \leq d+1\}, \\ \Theta_1 &= \{(1, j) \in \Theta \mid 0 \leq j \leq d\}, \\ \Theta_2 &= \{(2, j) \in \Theta \mid 0 \leq j \leq d-1\}, \end{aligned}$$

Each row has an order in which  $(i, j) \leq (i, j')$  if and only if  $j \leq j'$ .

Now we can redefine a *short Gelfand-Tsetlin pattern* to be an integer valued function  $\mathfrak{t}$  on the substrate, subject to the conditions that we have already stated. Thus the pattern (8) corresponds to the function on  $\Theta$  such that  $l_i = \mathfrak{t}(0, i)$ ,  $a_i = \mathfrak{t}(1, i)$  and  $b_i = \mathfrak{t}(2, i)$ . The  $\Gamma$  and  $\Delta$  arrays then become functions on  $\Theta_B$  (the bottom and middle rows) in the same way. Specifying the circled and boxed elements just means specifying subsets of  $\Theta_B$ .

Now the vertices of the cartoon will be the elements of  $\Theta$ , and we have only to define the edges. With  $\mathfrak{t}$  as in (8) we connect  $(1, i)$  to  $(0, i)$  if either  $i = 0$  or  $l_i \leq b_{i-1}$ , and we connect  $(1, i)$  to  $(2, i-1)$  if  $i > 1$  and  $b_{i-1} \leq l_i$ . Furthermore we connect

$(1, i)$  to  $(0, i + 1)$  if either  $i = d$  or if  $i < d$  and  $l_{i+1} \geq b_i$ , and we connect  $(1, i)$  to  $(2, i)$  if  $i < d$  and  $b_i \geq l_{i+1}$ . For example, consider the short pattern of rank 5:

$$\mathbf{t} = \begin{Bmatrix} 23 & 15 & 12 & 5 & 2 & 0 \\ & 20 & 12 & 5 & 4 & 2 \\ & & 14 & 9 & 5 & 3 \end{Bmatrix} \quad (28)$$

It is convenient to draw the cartoon as a graph on top of the array representing  $\mathbf{t}$ , as follows:

$$\begin{array}{ccccccccc} 23 & & 15 & & 12 & & 5 & & 2 & & 0 \\ & \diagdown & & \diagup & & \diagdown & & \diagup & & \diagdown & & \diagup \\ & & 20 & & 12 & & 5 & & 4 & & 2 \\ & & & \diagdown & & \diagup & & \diagdown & & \diagup & & \\ & & & & 14 & & 9 & & 5 & & 3 \\ & & & & & & & & & & & \end{array} \quad (29)$$

- The cartoon depends only on the top and bottom rows of  $\mathbf{t}$ , so it is really a function of the prototype  $\mathfrak{S}$  to which  $\mathbf{t}$  belongs.
- The cartoon encodes the relationship between  $\mathbf{t}$  and  $\mathbf{t}'$ . Indeed, suppose that the cartoon has a subgraph  $x \text{---} y \text{---} z$  where  $y$  is in the middle row,  $x$  and  $z$  are each in either the top or bottom row, with  $x$  is to the left of  $y$  and  $z$  is to the right. Then in  $\mathbf{t}'$ ,  $y$  is replaced by  $x + z - y$ .

For example, if  $\mathbf{t}$  is given by (28), then the cartoon (29) tells us how to compute

$$\mathbf{t}' = \begin{Bmatrix} 23 & 15 & 12 & 5 & 2 & 0 \\ & 18 & 14 & 9 & 4 & 0 \\ & & 14 & 9 & 5 & 3 \end{Bmatrix};$$

the middle row entries are  $18 = 23 + 15 - 20$ ,  $14 = 12 + 14 - 12$ ,  $9 = 5 + 9 - 5$ ,  $4 = 5 + 3 - 4$  and  $0 = 2 + 0 - 2$ .

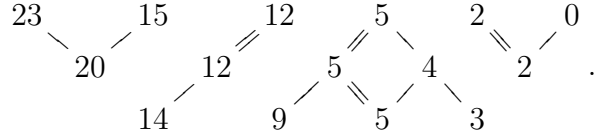
The connected components of the cartoon are called *episodes*. These may be arranged in an order  $\mathcal{E}_1, \dots, \mathcal{E}_N$  so that if  $i < j$  and  $\alpha \in \mathcal{E}_i$ ,  $\beta \in \mathcal{E}_j$ , and if  $\alpha$  and  $\beta$  are in the same row of  $\Theta$ , then  $\alpha < \beta$ . With this partial order, if  $\alpha \in \mathcal{E}_i$ ,  $\beta \in \mathcal{E}_j$  then  $\mathbf{t}(\alpha) > \mathbf{t}(\beta)$  regardless of whether or not  $\alpha$  and  $\beta$  are in the same row.

We call short pattern (8) *resonant* at  $i$  if  $l_{i+1} = b_i$ . A *resonance* of order  $k$  is a sequence  $R = \{i, i + 1, \dots, i + k - 1\}$  such that  $\mathbf{t}$  is resonant at each  $j \in R$ ; the sequence must be maximal with this property, so that  $\mathbf{t}$  is not resonant at  $i - 1$  or  $i + k$ . In the example (28), a resonance at 2 can be recognized from the cartoon by the diamond shape between  $l_3 = b_2 = 5$ .

We will next describe another kind of diagram related to the cartoon in which we mark certain edges with double bonds, and box and circle certain vertices. We will refer to the diagram in which the bonded edges and circled vertices are marked as the *bond-marked* cartoon. See (30) and (31) below for examples.

- Unlike the cartoon, the bond-marked cartoon really depends on  $\mathbf{t}$ , not just on its prototype.
- The bond-marked cartoon is useful since the circling and boxing of the  $\Gamma$  and  $\Delta$  arrays can be read off from it.

The edge joining  $\alpha, \beta \in \Theta$  will be called *distinguished* if  $\mathbf{t}(\alpha) = \mathbf{t}(\beta)$ . In representing the bond-marked cartoon graphically we will mark the distinguished edges by double bonds, which may be read as equal signs. Thus in the example (28), the cartoon of  $\mathbf{t}$  becomes



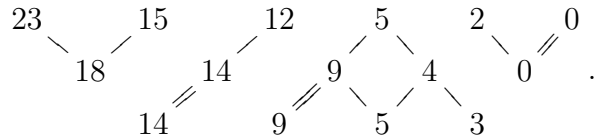
(We have drawn this labeling on top of the  $\mathbf{t}$  array, but ultimately we will draw it on top of the  $\Gamma$  or  $\Delta$  arrays.)

We observe that while the original bond-unmarked cartoon only depends on the pattern prototype  $\mathfrak{S}$  to which  $\mathbf{t}$  belongs, this diagram does depend on  $\mathbf{t}$ . In particular,  $\mathbf{t}$  and  $\mathbf{t}'$  no longer have the same cartoon, since the double bonds move under the involution  $\mathbf{t} \mapsto \mathbf{t}'$ . However the rule is quite simple:

**Lemma 1** *Suppose the bond-marked cartoon of  $\mathbf{t}$  has a subgraph of the form  $x - z = z$ , where the first  $z$  is in the middle row, so that  $x$  and the second  $z$  are in the top or bottom row. Then in the bond-marked cartoon the double bond moves to the other edge, so the bond-marked cartoon of  $\mathbf{t}'$  contains a subgraph  $x = x - z$ .*

**Proof** Immediate from the definitions. □

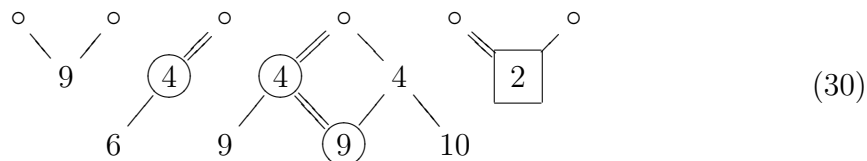
In this example, the bond-marked cartoon of  $\mathbf{t}'$  is



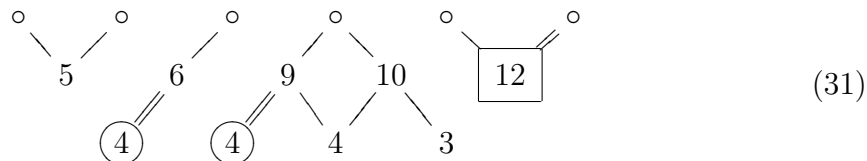
As we have already indicated, the cartoon is very useful when superimposed on the  $\Gamma$  and  $\Delta'$  arrays, where  $\Gamma = \Gamma_{\mathbf{t}}$  and  $\Delta' = \Delta_{\mathbf{t}'}$ . Since these arrays have only two rows, we add a third row at the top. We will also box and circle certain entries, by a convention that we will explain after giving an example. Thus in this example

$$\Gamma = \left\{ \begin{array}{ccccc} 9 & 4 & 4 & 4 & 2 \\ & 6 & 9 & 9 & 10 \\ & & & & \end{array} \right\}, \quad \Delta' = \left\{ \begin{array}{ccccc} 5 & 6 & 9 & 10 & 12 \\ & 4 & 4 & 4 & 3 \\ & & & & \end{array} \right\}.$$

We superimpose the cartoon on these, representing  $\Gamma$  thus:



and  $\Delta'$  as



We've inserted a row of  $\circ$ 's in the top (0-th) row since the  $\Gamma$  and  $\Delta'$  arrays have first and second rows but no 0-th row; we supply these for the purpose of drawing the bond-marked cartoon. When the bond-marked cartoon is thus placed on top of the  $\Gamma$  and  $\Delta'$  arrays, the circling and boxing conventions can be conveniently understood.

- In the first row of  $\Gamma$  or the second row of  $\Delta'$  we circle an entry if a double bond is above it and to the right. We box an entry if a double bond is above it and to the left. Thus:



- In the second row of  $\Gamma$  or the first row of  $\Delta'$  we circle an entry if a double bond is above it and to the left. We box an entry if a double bond is above it and to the right. Thus:



Now we have the basic language that will allow us to prove the reduction to the totally resonant case.

## 5 The Snake Lemma

The Snake Lemma that we are referring to was stated without proof in [7]. Here we will recall it, prove it, and use it to prove the statement made in the introduction, that (16) is “often” true.

By an *indexing* of the  $\Gamma$  array we mean a bijection

$$\phi : \{1, 2, \dots, 2d + 1\} \longrightarrow \Theta_B.$$

With such an indexing in hand, we will denote  $\Gamma_{\mathbf{t}}(\alpha)$  by  $\gamma_k(\mathbf{t})$  or just  $\gamma_k$  if  $\alpha = \phi(k)$  corresponds to  $k$ . Thus

$$\{\gamma_1, \gamma_2, \dots, \gamma_{2d+1}\} = \{\Gamma(\alpha) \mid \alpha \in \Theta_B\}.$$

We will also consider an indexing  $\psi$  of the  $\Delta'$  array, and we will denote  $\Delta'(\alpha)$  by  $\delta'_k$  if  $\alpha = \psi(k)$ . It will be convenient to extend the indexings by letting  $\gamma_0 = \gamma_{2d+2} = 0$  and  $\delta'_0 = \delta'_{2d+2} = 0$ .

**Proposition 2 (Snake Lemma)** *There exist indexings of the  $\Gamma$  and  $\Delta'$  arrays such that*

$$\delta'_k = \begin{cases} \gamma_k & \text{if } k \text{ is even,} \\ \gamma_k + \gamma_{k-1} - \gamma_{k+1} & \text{if } k \text{ is odd.} \end{cases} \quad (32)$$

*If  $i \in \{1, 2, \dots, 2d + 2\}$ , and if  $\phi(i) \in \mathcal{E}_k$ , then  $\psi(i) \in \mathcal{E}_k$  also. Moreover if  $\phi(j), \psi(j) \in \mathcal{E}_l$  and  $k < l$  then  $i < j$ .*

Before we prove this, let us confirm it in the specific example at hand. With  $\Gamma$  and  $\Delta'$  as in (30) and (31), we may take the correspondence as follows:

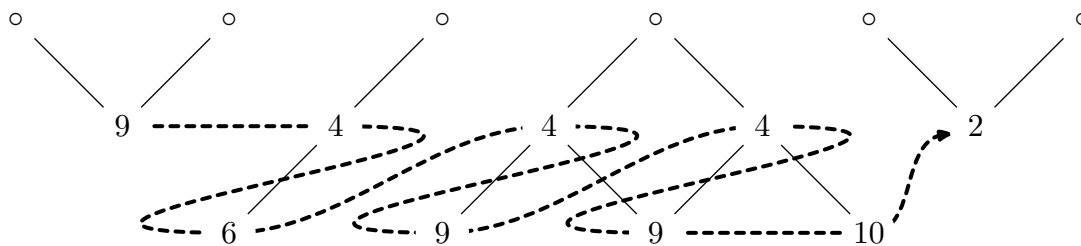
$k$	0	1	2	3	4	5	6	7	8	9	10
$(i, j)$ in the $\Gamma$ ordering		(1, 0)	(1, 1)	(2, 0)	(1, 2)	(2, 1)	(1, 3)	(2, 2)	(2, 3)	(1, 4)	
episode		$\mathcal{E}_1$	$\mathcal{E}_2$	$\mathcal{E}_2$	$\mathcal{E}_3$	$\mathcal{E}_3$	$\mathcal{E}_3$	$\mathcal{E}_3$	$\mathcal{E}_3$	$\mathcal{E}_4$	
$\gamma_k$	0	9	4	6	4	9	4	9	10	2	0
$(i, j)$ in the $\Delta'$ ordering		(1, 0)	(2, 0)	(1, 1)	(2, 1)	(1, 2)	(2, 2)	(2, 3)	(1, 3)	(1, 4)	
$\delta'_k$	0	5	4	6	4	9	4	3	10	12	0
episode		$\mathcal{E}_1$	$\mathcal{E}_2$	$\mathcal{E}_2$	$\mathcal{E}_3$	$\mathcal{E}_3$	$\mathcal{E}_3$	$\mathcal{E}_3$	$\mathcal{E}_3$	$\mathcal{E}_4$	

(33)

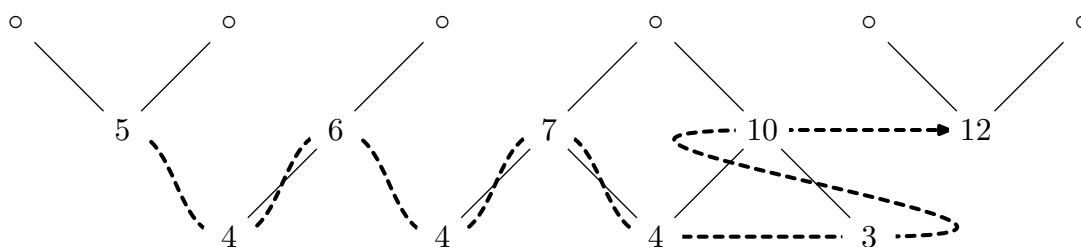
The meaning of the last assertion of the Snake Lemma is that each indexing visits the episodes of the sequence in order from left to right, and after it is finished with an episode, it moves on to the next with no skipping around. Thus both indexings must be in the same episode.

The reason for the name ‘‘Snake Lemma’’ is that if one connects the nodes of  $\Theta_B$  in the indicated orderings, a pair of ‘‘snakes’’ becomes visible. Thus in the example

(33) the paths will look as follows. The  $\Gamma$  indexing is represented:



and the  $\Delta'$  indexing:



- The proof will provide a particular description of the pair of snakes; in applying the Snake Lemma will sometimes need this particular description. We will describe the pair of indexings, or “snakes,” as *canonical* if they are produced by the method described in the proof, which is expressed in Table 1 below. Thus we implicitly use the proof as well as the statement of the Lemma.
- If there are resonances, there will be more than one possible pair of snakes. (Indeed, the reader will find another way of drawing the snakes in the preceding example.) These will be obtained through a process of *specialization* that will be described in the proof. Any one of these pairs of snakes will be described as canonical.

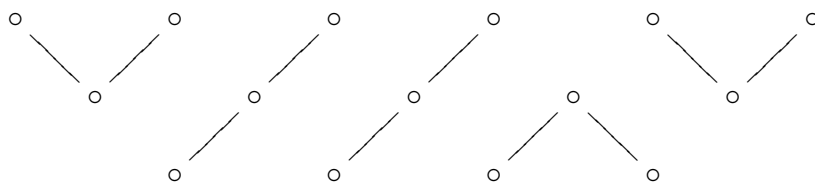
**Proof** For this proof, double bonds are irrelevant, and we will work with the bond-unmarked cartoon. Thus both  $\Gamma$  and  $\Delta'$  are again represented by the same cartoon, which in the example (28) was the cartoon (29). Resonances are a minor complication, which we eliminate as follows. We divide the cartoon into *panels*, each being of one five types:

$t$	$T$	$B$	$b$	$R$

The first panel is always of type  $t$  and the last one of type  $b$ . We call a cartoon *simple* if it contains no panels of type  $R$ .

The panel type  $R$  occurs at each resonance. Including it in our discussion would unnecessarily increase the number of cases to be considered, so we resolve each resonance by arbitrarily replacing each  $R$  by either a  $T$  or  $B$ . This will produce a simple cartoon. We refer to this process as *specialization*.

For example the cartoon (29) corresponds to the word  $tBBRTb$ , meaning that these panels appear in sequence from left to right. We replace the resonant panel  $R$  arbitrarily by either  $T$  or  $B$ ; for example if we choose  $B$  we obtain the simple cartoon:



Now, from the simple cartoon we may describe the algorithm for finding the pair of snakes, that is, the  $\Gamma$  and  $\Delta'$  indexings. Each connected component (episode) in the simple cartoon has three vertices, the middle one being in the second row. We may classify these episodes into four classes as follows:

Class I	Class II	Class III	Class IV

Now we can describe the snakes. For each episode, we select a path from Table 1. The nodes labeled  $\star$  will turn out to be indexed by even integers, and the nodes labeled  $\bullet$  will be indexed by odd integers. We've subscripted the  $\star$ 's to indicate which entries are corresponding in the  $\Gamma$  and  $\Delta'$ . A ? means that the information at hand does not determine whether the entry will be even or odd in the indexing, so we do not attempt to assign it a  $\star$  or  $\bullet$ .

There are modifications at the left and right edges of the pattern: for example, if the first two panels are  $tB$  then the first connected component is of Class II, and the left parts of the  $\Gamma$  and  $\Delta'$  indexings indicated in the table are missing. Thus we

	$\Gamma$ labeling	$\Delta'$ labeling
Class I		
Class II		
Class III		
Class IV		

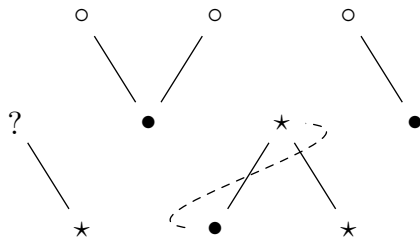
Table 1: Snake taxonomy.

have a modification of the Class II pattern that we call  $II_t$ .

	$\Gamma$ indexing	$\Delta'$ indexing
Class $II_t$		

Similarly there are Classes  $\text{III}_t$ ,  $\text{II}_b$  and  $\text{IV}_b$  that can occur at the left or right edge of the pattern. In every case these are obtained by simply deleting part of the corresponding pattern, and we will not enumerate these for this reason.

It is necessary to see that these paths are assigned consistently. For example, suppose that the cartoon contains consecutive panels  $TBT$ . Inside the resulting configuration are both a Class II connected component and a Class I component. Referring to Table 1, both these configurations mandate the following dashed line in the  $\Gamma$  diagram.



This sort of consistency must be checked in eight cases, which we leave to the reader. Then it is clear that splicing together the segments prescribed this way gives a consistent pair of snakes, and the indexings can be read off from left to right.

It remains, however, for us to prove (32). This is accomplished by four Lemmas. There are many things to verify; we will do one and leave the rest to the reader.

**Lemma 2** *If the  $j$ -th connected component is of Class I, then*

$$\begin{aligned}\Delta'(1, j) &= \Gamma(2, j) \\ \Delta'(2, j-1) &= \Gamma(1, j) \\ \Delta'(2, j) &= \Gamma(2, j-1) + \Gamma(1, j) - \Gamma(2, j).\end{aligned}$$

This asserts a part of (32), namely the equality  $\delta'_k = \gamma_k$  for the vertices labeled  $\star_1$  and  $\star_2$  in Table 1, and the equality  $\delta'_k = \gamma_k + \gamma_{k-1} - \gamma_{k+1}$  for the unstarred vertex in the connected component.

We prove that  $\Delta'(1, j) = \Gamma(2, j)$ . With  $a'_i$  defined by (11) and (12) we have

$$\Delta'(1, j) = \left( \sum_{i \leq j} l_i \right) - \left( \sum_{i \leq j} a'_i \right).$$

Our assumption that the  $j$ -th connected component is of Class I means that  $l_j \geq b_{j-1}$

and that  $l_{j+1} \geq b_j$ , so  $a'_j = b_{j-1} + b_j - a_j$ . Moreover, since  $l_j \geq b_{j-1}$  we have

$$\begin{aligned} \sum_{i < j} [\min(l_i, b_{i-1}) + \max(l_{i+1}, b_i)] &= \\ \max(l_j, b_{j-1}) + \sum_{i < j} [\min(l_i, b_{i-1}) + \max(l_i, b_{i-1})] &= \\ l_j + \sum_{i < j} (l_i + b_{i-1}) &= \sum_{i \leq j} l_i + \sum_{i \leq j-2} b_i. \end{aligned}$$

By (11) we therefore have

$$\sum_{i \leq j} a'_i = \sum_{i \leq j} l_i + b_i - a_i, \quad \Delta'(1, j) = \sum_{i \leq j} a_i - b_i = \Gamma(2, j).$$

We leave the remaining two statements to the reader.

**Lemma 3** *If the  $j$ -th connected component is of Class II, then*

$$\Delta'(1, j) = \Gamma(1, j) + \Gamma(2, j-1) - \Gamma(1, j+1).$$

**Lemma 4** *If the  $j$ -th connected component is of Class III, then*

$$\begin{aligned} \Delta'(1, j) &= \Gamma(2, j), \\ \Delta'(2, j) &= \Gamma(1, j) + \Gamma(2, j-1) - \Gamma(2, j). \end{aligned}$$

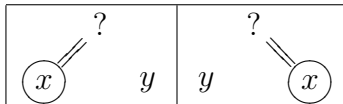
**Lemma 5** *If the  $j$ -th connected component is of Class IV, then*

$$\begin{aligned} \Delta'(2, j-1) &= \Gamma(1, j), \\ \Delta'(1, j) &= \Gamma(2, j-1) + \Gamma(1, j) - \Gamma(1, j+1). \end{aligned}$$

We leave the proofs of the last Lemmas to the reader. The assertions in (32) are contained in the Lemmas, for each  $\delta'_k$  whose corresponding vertex is in the given connected component. It must lie in the first (middle) or second row of the cartoon, which is why there are three identities for Class I, one for Class II and two for Classes III and IV. Thus (32) is proved for every  $\delta'_k$ . The final assertion, that the episodes of the cartoon are visited from left to right in order by both indexings, can be seen by inspection from Table 1.  $\square$

**Lemma 6 (Circling Lemma)** *Assume that  $\mathbf{t}$  is strict.*

(i) *Suppose that either of the following two configurations occurs in either  $\Gamma_{\mathbf{t}}$  or  $\Delta_{\mathbf{t}}$ . Then  $x = y$ .*



(ii) *If  $x$  occurs circled in either the  $\Gamma$  or  $\Delta$  array of a strict pattern  $\mathbf{t}$ , then either the same value  $x$  also occurs uncircled (and unboxed) at another location, or  $x = 0$ .*

**Proof** The first statement follows from the definition. To prove the second statement, we note that  $y = x$  is unboxed since the pattern is strict. If it is uncircled, (ii) is proved. If it is circled, we continue to the right (if  $y$  is to the right of  $x$ ) or to the left (if  $y$  is to the left of  $x$ ) until we come to an uncircled one. This can only fail if we come to the edge of the pattern. If this happens, then  $x = 0$ .  $\square$

## 6 Noncritical Resonances

We recall that a short pattern (8) is *resonant* at  $i$  if  $l_{i+1} = b_i$ . This property depends only on the associated prototype, so resonance is actually a property of prototypes. We also call a first (middle) row entry  $a_i$  *critical* if it is equal to one of its four *neighbors*, which are  $l_i, l_{i+1}, b_i$  and  $b_{i-1}$ . We say that the resonance at  $i$  is *critical* if either  $a_i$  or  $a_{i+1}$  is critical.

**Theorem 2** *Suppose that  $\mathbf{t}$  is a strict pattern with no critical resonances; then  $\mathbf{t}'$  is also strict with no critical resonances. Choose canonical indexings  $\gamma_i$  and  $\delta'_i$  as in the Snake Lemma (Proposition 2). Then either  $G_{\Gamma}(\mathbf{t}) = G_{\Delta}(\mathbf{t}) = 0$  or  $n|\gamma_i$ . In any case, we have*

$$G_{\Gamma}(\mathbf{t}) = G_{\Delta}(\mathbf{t}').$$

As an example, the pattern is called *superstrict* if the inequalities (13) and (14) are strict, that is, if

$$\min(l_j, b_{j-1}) > a_j > \max(l_{j+1}, b_j), \quad 0 < j < d, \quad (34)$$

$$l_1 > a_1 > \max(l_2, b_1), \quad \min(l_d, b_{d-1}) > a_d > l_{d+1}. \quad (35)$$

Thus if the patterns within a type are regarded as lattice points in a polytope, the superstrict patterns are the interior points. Again, the pattern (or prototype) is called *nonresonant* if there are no resonances. The theorem is clearly applicable if  $\mathbf{t}$  is either superstrict or nonresonant.

**Proof** To see that  $\mathbf{t}'$  is strict, let  $a_i, b_i, l_i$  and  $a'_i$  be as in (8) and (10). If  $\mathbf{t}'$  is not strict, we must have  $a'_i = a'_{i-1}$  for some  $i$ , and it is easy to see that this implies that  $l_i = b_{i-1}$ , and that  $\mathbf{t}$  has a critical resonance at  $i$ . It is also easy to see that if  $\mathbf{t}'$  has a critical resonance at  $i$  so does  $\mathbf{t}$ .

Choose a pair of canonical indexings of  $\Gamma = \Gamma_{\mathbf{t}}$  and  $\Delta' = \Delta_{\mathbf{t}'}$ . Our first task is to show that either  $G_{\Gamma}(\mathbf{t}) = G_{\Delta}(\mathbf{t}') = 0$  or  $n|\gamma_i$  for all even  $i$ . It is easy to see that  $\gamma_i$  and  $\delta'_i$  are not boxed, since if it were, examination of every case in Table 1 shows that it would be at the terminus of a double bond in the bond-marked cartoon that is not one of the marked bonds in the figures. This could conceivably happen since in the proof of Proposition 2 we began by replacing the cartoon by a simple cartoon, a process that can involve discarding some parallel pairs of the bonds; however it would force  $\gamma_i$  (or  $\delta'_i$ ) to be a neighbor of a critical resonance, and we are assuming that  $\mathbf{t}$  has no critical resonances.

Suppose that  $\gamma_i$  is not circled ( $i$  even). Then  $G_{\Gamma}(\mathbf{t})$  is a multiple of  $h(\gamma_i)$ , which vanishes unless  $n|\gamma_i$ . If  $\gamma_i$  is circled, we must argue differently. By Lemma 6, either the same value  $\gamma_i$  occurs uncircled and unboxed somewhere in the  $\Gamma$  array, in which case  $G_{\Gamma}(\mathbf{t})$  is again a multiple of  $h(\gamma_i)$ , or  $\gamma_i = 0$ . Since  $n|\gamma_i$  if  $\gamma_i = 0$  the conclusion that  $G_{\Gamma}(\mathbf{t}) = 0$  or  $n|\gamma_i$  is proved. Since  $\gamma_i = \delta'_i$  when  $n$  is even, we may also conclude that  $G_{\Delta}(\mathbf{t}') = 0$  unless the  $\gamma_i$  ( $i$  even) are all divisible by  $n$ .

We assume for the remainder of the proof that  $n|\gamma_i$  when  $i$  is even. Let us denote

$$\tilde{\gamma}_i = \begin{cases} q^{\gamma_i} & \text{if } \gamma_i \text{ is circled in the } \Gamma \text{ indexing,} \\ g(\gamma_i) & \text{if } \gamma_i \text{ is boxed in the } \Gamma \text{ indexing,} \\ h(\gamma_i) & \text{otherwise,} \end{cases}$$

with  $\tilde{\delta}'_i$  defined similarly. Thus

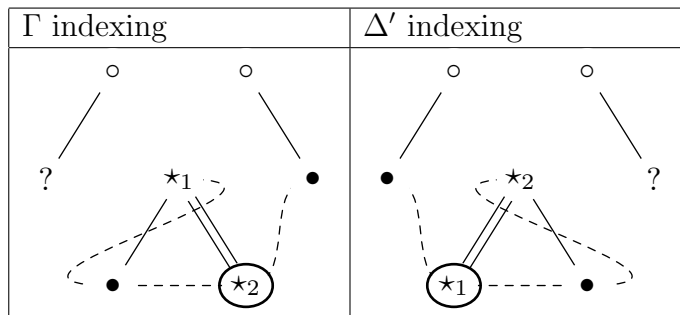
$$G_{\Gamma}(\mathbf{t}) = \prod \tilde{\gamma}_i, \quad G_{\Delta}(\mathbf{t}') = \prod \tilde{\delta}'_i.$$

We next show that

$$\prod_{i \text{ even}} \tilde{\gamma}_i = \prod_{i \text{ even}} \tilde{\delta}'_i. \quad (36)$$

Since  $\gamma_i = \delta'_i$  when  $i$  is even, and since as we have noted these entries are never boxed, the only way this could fail is if one of  $\tilde{\gamma}_i$  and  $\tilde{\delta}'_i$  is circled and the other not. We look at the connected component in the bond-unmarked cartoon containing  $\tilde{\gamma}_i$ . In Table 1, this entry is starred and must correspond to one of  $\star_1, \star_2, \star_6$  or  $\star_7$ . (Since the snake is obtained by splicing pieces together different pieces of Table 1 it may also appear  $\star_3, \star_4, \star_5$  or  $\star_8$ .) If it is  $\star_6$ , then it is circled in the  $\Gamma$  indexing if and only if the bond above it is doubled, and by Lemma 1 the bond above  $\star_6$  in

the  $\Delta'$  indexing is also doubled, so  $\star_6$  is circled in both indexings; and similarly with  $\star_7$ . Turning to the Class I components, it is impossible for  $\star_1$  to be circled in the  $\Gamma$  indexing, since this would imply a critical resonance; and  $\star_2$  is never circled in the  $\Delta'$  indexing for the same reasoning. Nevertheless it is possible for  $\star_2$  to be circled in the  $\Gamma$  indexing but not the  $\Delta'$  indexing. In this case, Lemma 1 shows that  $\star_2$  is starred in the  $\Delta'$  indexing but not the  $\Gamma$  indexing. This happens when the labeling of a Class I component looks like:



Thus if  $\star_1$  is the  $i$ -th vertex in both orderings we have

$$\tilde{\gamma}_i = h(\gamma_i), \quad \tilde{\gamma}_{i+2} = q^{\gamma_{i+2}}, \quad \tilde{\delta}'_i = q^{\delta'_i} = q^{\gamma_i}, \quad \tilde{\delta}'_{i+2} = h(\delta'_{i+2}) = h(\gamma'_{i+2}),$$

and it is still true that  $\tilde{\gamma}_i \tilde{\gamma}_{i+2} = \tilde{\delta}'_i \tilde{\delta}'_{i+2}$ . This proves (36).

Now we prove

$$\prod_{i \text{ odd}} \tilde{\gamma}_i = \prod_{i \text{ odd}} \tilde{\delta}'_i. \quad (37)$$

When  $i$  is odd, it follows from Lemma 1 that  $\gamma_i$  is circled or boxed in the  $\Gamma$  indexing if and only if  $\delta'_i$  is. Using (32), and remembering that since  $i - 1$  and  $i + 1$  are even we are now assuming  $\gamma_{i-1}$  and  $\gamma_{i+1}$  are multiples of  $n$ , we obtain

$$\tilde{\delta}'_i = q^{\gamma_{i-1} - \gamma_{i+1}} \tilde{\gamma}_i.$$

Thus taking the product over odd  $i$ , the powers of  $q$  will cancel in pairs, giving (37). Combining this with (36), the theorem is proved.  $\square$

There is another important case where (16) is true. This is case where the pattern  $\mathbf{t}$  is *stable*. We say that  $\mathbf{t}$  is *stable* if each  $a_i$  equals either  $l_i$  or  $l_{i+1}$ , and each  $b_i$  equals either  $a_i$  or  $a_{i+1}$ . Thus every element of the  $\Gamma$  array is either circled or boxed. If this is true then it follows from Lemma 1 that  $\mathbf{t}'$  is also stable. Theorem 2 does not apply to stable patterns since they usually have critical resonances.

**Theorem 3** *Suppose that  $\mathbf{t}$  is stable. Then  $G_\Gamma(\mathbf{t}) = G_\Delta(\mathbf{t}')$ .*

**Proof** It is easy to see that every element of the  $\Gamma$  and  $\Delta'$  arrays is either circled or boxed, and that the circled entries are precisely the ones that equal zero. As we will explain, the boxed elements are precisely the same for the  $\Gamma$  and  $\Delta'$  arrays.

Let  $S$  be the set of elements of the top row of  $\mathfrak{t}$ . Between the top row and the row below it, one element is omitted; call it  $a$ . Between this row and the next, another element is omitted; call this  $b$ . In  $\mathfrak{t}'$  the same two elements are dropped, but in the reverse order. The boxed entries that appear in  $\Gamma$  are

first row:	$\{x - a   x \in S, x > a\}$
second row:	$\{b - x   x \in S, x < b, x \neq a\}$

The boxed entries that appear in  $\Delta'$  are:

first row:	$\{b - x   x \in S, x > b\}$
second row:	$\{x - a   x \in S, x > a, x \neq b\}$

The entry  $a - b$  appears in both cases only if  $a > b$ . The statement is now clear.  $\square$

## 7 Types

We now divide the prototypes into much smaller units that we call *types*. We fix a top and bottom row, and therefore a cartoon. For each episode  $\mathcal{E}$  of the cartoon, we fix an integer  $k_{\mathcal{E}}$ . Then the set  $\mathfrak{S}$  of all short Gelfand-Tsetlin patterns (8) with the given top and bottom rows such that for each  $\mathcal{E}$

$$\sum_{\alpha \in \Theta_1 \cap \mathcal{E}} \mathfrak{t}(\alpha) = k_{\mathcal{E}} \quad (38)$$

is called a *type*. Thus two patterns are in the same type if and only if they have the same top and bottom rows (and hence the same cartoon), and if the sum of the first (middle) row elements in each episode is the same for both patterns.

Let us choose  $\Gamma$  and  $\Delta'$  indexings as in the Snake Lemma (Proposition 2). With notations as in that Lemma, and  $\mathcal{E}$  a fixed episode of the corresponding cartoon, there exist  $k$  and  $l$  such that  $\phi(i) \in \mathcal{E}$  and  $\psi(i) \in \mathcal{E}$  precisely when  $k \leq i \leq l$ . let

$$L_{\mathcal{E}} = \begin{cases} k & \text{if } k \text{ is even,} \\ k - 1 & \text{if } k \text{ is odd,} \end{cases} \quad R_{\mathcal{E}} = \begin{cases} l & \text{if } l \text{ is even,} \\ l + 1 & \text{if } l \text{ is odd.} \end{cases}$$

Then Proposition 2 implies that

$$\sum_{i=k}^l \delta_i(\mathfrak{t}') = \left( \sum_{i=k}^l \gamma_i(\mathfrak{t}) \right) + \gamma_{L_{\mathcal{E}}}(\mathfrak{t}) - \gamma_{R_{\mathcal{E}}}(\mathfrak{t}), \quad (39)$$

for all elements of the type. We recall that our convention was that  $\gamma_0 = \gamma_{2d+2} = 0$ . We take  $L_{\mathcal{E}} = 0$  for the first (leftmost) cartoon and  $R_{\mathcal{E}} = 2d + 2$  for the last episode.

We may classify the possible episodes into four classes generalizing the classification in Table 1, and indicate in each case the locations of  $\gamma_{L_{\mathcal{E}}}$  and  $\gamma_{R_{\mathcal{E}}}$  in the  $\Gamma$  arrays, which may be checked by comparison with Table 1. Indeed, it must be remembered that in that proof, every panel of type  $R$  is replaced by one of type  $T$  or type  $B$ . Whichever choice is made, Table 1 gives the same location for  $L_{\mathcal{E}}$  and  $R_{\mathcal{E}}$ . The classification of the episode into one of four types is given in Table 2.

Class I	
Class II	
Class III	
Class IV	

Table 2: The four classes of episodes in  $\Gamma_{\mathfrak{t}}$ .

The location of  $\delta'_{L_{\mathcal{E}}}$  and  $\delta'_{R_{\mathcal{E}}}$  in the  $\Delta$  array of  $\mathfrak{t}'$  may also be read off from Table 1. The classification of the episode into one of four classes is given in Table 3.

**Proposition 3** *If  $\mathcal{F}$  is the episode that consecutively follows  $\mathcal{E}$ , then  $R_{\mathcal{E}} = L_{\mathcal{F}}$ . The values  $\gamma_{L_{\mathcal{E}}}(\mathfrak{t})$  and  $\gamma_{R_{\mathcal{E}}}(\mathfrak{t})$  are constant on each type. Moreover  $G_{\Gamma}(\mathfrak{t}) = G_{\Delta}(\mathfrak{t}') = 0$  for all patterns  $\mathfrak{t}$  in the type unless the  $\gamma_{L_{\mathcal{E}}}$  are divisible by  $n$ . In the  $\Gamma$  and  $\Delta'$  arrays,  $\gamma_{L_{\mathcal{E}}}$  and  $\delta'_{L_{\mathcal{E}}}$  may be circled or not, but never boxed.*

**Proof** From Tables 2 and 3, it is clear that  $R_{\mathcal{E}} = L_{\mathcal{F}}$  for consecutive episodes.

Class I	
Class II	
Class III	
Class IV	

Table 3: The four classes of episodes in  $\Delta_{\mathcal{U}}$ .

If  $\mathcal{E}$  is of Class I or Class IV, then we see that

$$\gamma_{L\mathcal{E}} = \sum_{\mathcal{F} \geq \mathcal{E}} \left( \sum_{\alpha \in \Theta_1 \cap \mathcal{F}} \mathbf{t}(\alpha) - \sum_{\alpha \in \Theta_0 \cap \mathcal{F}} \mathbf{t}(\alpha) \right),$$

where the notation means that we sum over all episodes to the right of  $\mathcal{E}$  (including  $\mathcal{E}$  itself). If  $\mathcal{E}$  is of Class II or III, we have

$$\gamma_{L\mathcal{E}} = \sum_{\mathcal{F} \leq \mathcal{E}} \left( \sum_{\alpha \in \Theta_1 \cap \mathcal{F}} \mathbf{t}(\alpha) - \sum_{\alpha \in \Theta_2 \cap \mathcal{F}} \mathbf{t}(\alpha) \right).$$

In either case, these formulas imply that  $\gamma_{L\mathcal{E}}$  is constant on the patterns of the type.

Given their described locations, the fact that  $\gamma_{L\mathcal{E}} = \delta'_{L\mathcal{E}}$  is never boxed in either the  $\Gamma$  or  $\Delta'$  arrays may be seen from the definitions.

We now show that  $G_\Gamma(\mathbf{t}) = G_\Delta(\mathbf{t}') = 0$  unless  $n|\gamma_{L_\mathcal{E}}$ . Indeed, it follows from an examination of the locations of  $\gamma_{L_\mathcal{E}}$  in the  $\Gamma$  arrays and in the  $\Delta'$  arrays (where it appears as  $\delta'_{L_\mathcal{E}}$ ) that it is unboxed in both  $\Gamma$  and  $\Delta'$ . If it is uncircled in the  $\Gamma$  array, then  $G_\Gamma(\mathbf{t})$  is divisible by  $h(\gamma_{L_\mathcal{E}})$ , hence vanishes unless  $n|\gamma_{L_\mathcal{E}}$ . If it is circled, then we apply Lemma 6 to conclude that the same value appears somewhere else uncircled and unboxed, unless  $\gamma_{L_\mathcal{E}} = 0$  (which is divisible by  $n$ ), which again forces  $G_\Gamma(\mathbf{t}) = 0$  if  $n \nmid \gamma_{L_\mathcal{E}}$ ; and similarly for  $G_\Delta(\mathbf{t}') = 0$ .  $\square$

- Due to this result, we may impose the assumption that  $n|\gamma_{L_\mathcal{E}}$  for every episode. This assumption is in force for the rest of the paper.

Now let  $\mathcal{E}_1, \dots, \mathcal{E}_N$  be the episodes of the cartoon arranged from left to right, and let  $k_i = k(\mathcal{E}_i)$ . By a *local pattern* on  $\mathcal{E}_i$  subordinate to  $\mathfrak{S}$  we mean an integer-valued function on  $\mathcal{E}_i$  that can occur as the restriction of an element of  $\mathfrak{S}$  to  $\mathcal{E}_i$ . Its top and bottom rows are thus the restrictions of the given top rows, and it follows from the definition of the episode that if  $(0, t)$  and  $(2, t - 1)$  are both in  $\mathcal{E}_i$  then  $\mathbf{t}(0, t) = \mathbf{t}(2, t - 1)$ ; that is, if both an element of the top row and the element of the bottom row that is directly below it are in the same episode, then  $\mathbf{t}$  has the same value on both, and patterns in the type are resonant at  $t$ . The local pattern is subject to the same inequalities as a short pattern, and by (39) the sum of its first (middle) row elements must be  $k_i$ . Let  $\mathfrak{S}_i$  be the set of local patterns subordinate to  $\mathfrak{S}$ . We call  $\mathfrak{S}_i$  a *local type*.

**Lemma 7** *A pattern is in  $\mathfrak{S}$  if and only if its restriction to  $\mathcal{E}_i$  is in  $\mathfrak{S}_i$  for each  $i$  and so we have a bijection*

$$\mathfrak{S} \cong \mathfrak{S}_1 \times \dots \times \mathfrak{S}_N.$$

**Proof** This is obvious from the definitions, since the inequalities (38) for the various episodes are independent of each other.  $\square$

Now if  $\mathbf{t}$  is a short pattern let us define for each episode  $\mathcal{E}$

$$\begin{aligned} G_\Gamma^\mathcal{E}(\mathbf{t}) &= \prod_{\alpha \in \mathcal{E} \cap \Theta_B} \begin{cases} g(\alpha) & \text{if } \alpha \text{ is boxed in } \Gamma_{\mathbf{t}}, \\ q^\alpha & \text{if } \alpha \text{ is circled in } \Gamma_{\mathbf{t}}, \\ h(\alpha) & \text{otherwise,} \end{cases} \\ G_\Delta^\mathcal{E}(\mathbf{t}) &= \prod_{\alpha \in \mathcal{E} \cap \Theta_B} \begin{cases} g(\alpha) & \text{if } \alpha \text{ is boxed in } \Delta_{\mathbf{t}}, \\ q^\alpha & \text{if } \alpha \text{ is circled in } \Delta_{\mathbf{t}}, \\ h(\alpha) & \text{otherwise,} \end{cases} \end{aligned} \quad (40)$$

provided  $\mathbf{t}$  is *locally strict* at  $\mathcal{E}$ , by which we mean that if  $\alpha, \beta \in \mathcal{E} \cap \Theta_1$  and  $\alpha$  is to the left of  $\beta$  then  $\mathbf{t}(\alpha) > \mathbf{t}(\beta)$ . If  $\mathbf{t}$  is not locally strict, then we define  $G_\Gamma^\mathcal{E}(\mathbf{t}) = G_\Delta^\mathcal{E}(\mathbf{t}) = 0$ .

**Proposition 4** *Suppose that  $n|\gamma_{L\mathcal{E}}$  for every episode. Assume also that for each  $\mathfrak{S}_i$  we have*

$$\sum_{\mathfrak{t}_i \in \mathfrak{S}_i} G_{\Delta}^{\mathcal{E}_i}(\mathfrak{t}'_i) = q^{\gamma_{L\mathcal{E}_i} - \gamma_{R\mathcal{E}_i}} \sum_{\mathfrak{t}_i \in \mathfrak{S}_i} G_{\Gamma}^{\mathcal{E}_i}(\mathfrak{t}_i). \quad (41)$$

Then

$$\sum_{\mathfrak{t} \in \mathfrak{S}} G_{\Delta}(\mathfrak{t}') = \sum_{\mathfrak{t} \in \mathfrak{S}} G_{\Gamma}(\mathfrak{t}). \quad (42)$$

Two observations are implicit in the statement of equation (41).

- Since by its definition  $G_{\Gamma}^{\mathcal{E}_i}(\mathfrak{t})$  depends only on the restriction  $\mathfrak{t}_i$  of  $\mathfrak{t}$  to  $\mathfrak{S}_i$ , we may write  $G_{\Gamma}^{\mathcal{E}_i}(\mathfrak{t}_i)$  instead of  $G_{\Gamma}^{\mathcal{E}_i}(\mathfrak{t})$ , and this is well-defined.
- The statement uses the fact that  $\gamma_{L\mathcal{E}}(\mathfrak{t})$  and  $\gamma_{R\mathcal{E}}(\mathfrak{t})$  are constant on the type, since otherwise  $q^{\gamma_{L\mathcal{E}_i} - \gamma_{R\mathcal{E}_i}}$  would be inside the summation.

**Proof** If  $\mathfrak{t}_i \in \mathfrak{S}_i$  is the restriction of  $\mathfrak{t} \in \mathfrak{S}$ , we have

$$\sum_{\mathfrak{t} \in \mathfrak{S}} G_{\Gamma}(\mathfrak{t}) = \prod_i \sum_{\mathfrak{t}_i \in \mathfrak{S}_i} G_{\Gamma}^{\mathcal{E}_i}(\mathfrak{t}_i) = \prod_i q^{\gamma_{L\mathcal{E}_i} - \gamma_{R\mathcal{E}_i}} \sum_{\mathfrak{t}_i \in \mathfrak{S}_i} G_{\Delta}^{\mathcal{E}_i}(\mathfrak{t}'_i).$$

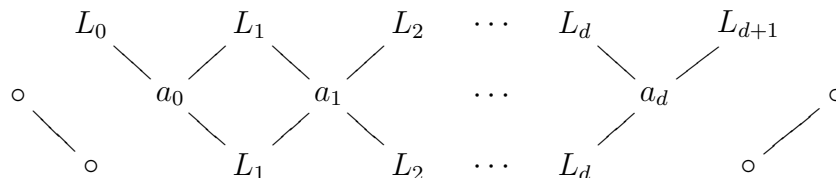
By Proposition 3, the factors  $q^{\gamma_{L\mathcal{E}_i} - \gamma_{R\mathcal{E}_i}}$  cancel each other  $R_{\mathcal{E}_i} = L_{\mathcal{E}_{i+1}}$ , and since our convention is that  $\gamma_0 = \gamma_{2d+2} = 0$ . Thus we obtain

$$\prod_i \sum_{\mathfrak{t}_i \in \mathfrak{S}_i} G_{\Delta}^{\mathcal{E}_i}(\mathfrak{t}'_i) = \sum_{\mathfrak{t} \in \mathfrak{S}} G_{\Delta}(\mathfrak{t}').$$

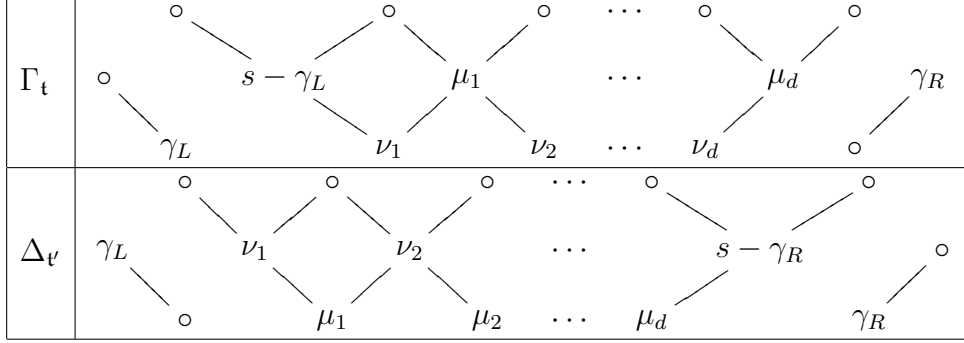
□

In the rest of the section we will fix an episode  $\mathcal{E} = \mathcal{E}_i$ , and let  $L = L_{\mathcal{E}}$  and  $R = R_{\mathcal{E}}$  to simplify the notation for the four remaining Propositions, which describe more precisely the relations between the  $\Gamma$  and  $\Delta'$  arrays within the episode  $\mathcal{E}$ .

**Proposition 5** *Let  $\mathfrak{t}$  be a short pattern whose cartoon contains the following Class II resonant episode  $\mathcal{E}$  of order  $d$ :*



Then there exist integers  $s, \mu_1, \nu_1, \mu_2, \nu_2, \dots, \mu_d, \nu_d$  such that  $\mu_i + \nu_i = s$  ( $i = 1, \dots, d$ ), and the  $\Gamma$  and  $\Delta'$  arrays are given in the following table.



The values  $s, \gamma_L$  and  $\gamma_R$  are constant on the type containing the pattern.

**Note:** If the episode  $\mathcal{E}$  occurs at the left edge of the cartoon, then our convention is that  $\gamma_L = \gamma_0 = 0$ , and if  $\mathcal{E}$  occurs at the right edge of the cartoon, then  $\gamma_R = \gamma_{2d+2} = 0$ . We would modify the picture by omitting  $\gamma_L$  or  $\gamma_R$  in these cases, but the proof below is unchanged.

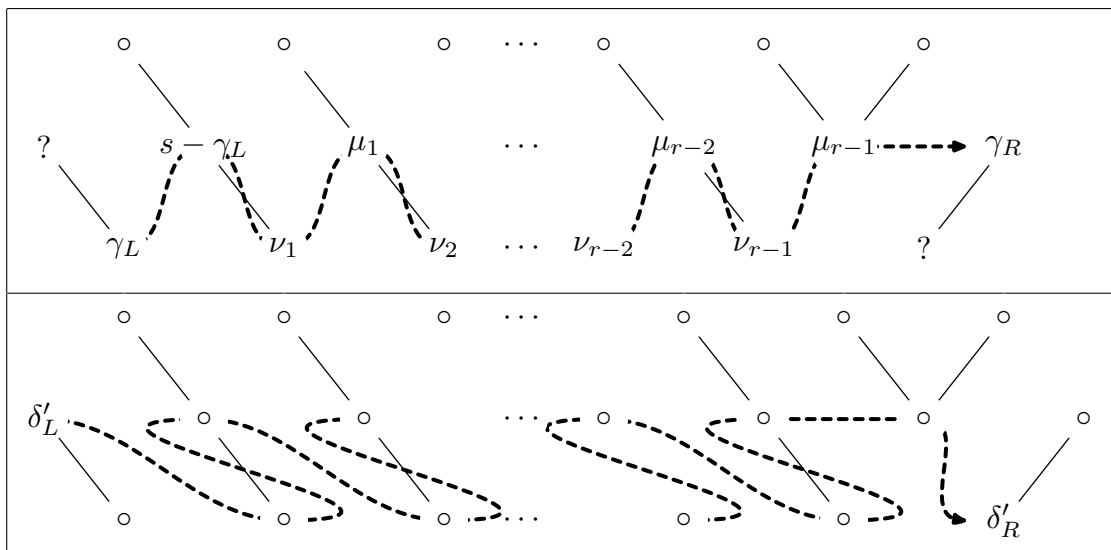
**Proof** Let  $\gamma_L = \gamma_{L_{\mathcal{E}}}$  and  $\gamma_R = \gamma_{R_{\mathcal{E}}}$  in the notation of the previous section, and let  $s, \mu_i, \nu_i$  be defined by their locations in the  $\Gamma$  array. Let  $s = \hat{s} + \gamma_R + \gamma_L$ ,  $\mu_i = \hat{\mu}_i + \gamma_R$  and  $\nu_i = \hat{\nu}_i + \gamma_L$ . It is immediate from the definitions that

$$\hat{s} = \sum_{j=0}^d (a_j - L_{j+1}), \quad \hat{\mu}_i = \sum_{j=i}^d (a_j - L_{j+1}), \quad \hat{\nu}_i = \sum_{j=0}^{i-1} (a_j - L_{j+1}).$$

From this it we see that  $\mu_i + \nu_i = s$  and  $\hat{\mu}_i + \hat{\nu}_i = \hat{s}$ .

In order to check the correctness of the  $\Delta'$  diagram, we observe that the resonance contains  $d$  panels of type  $R$ , each of which may be specialized to a panel of type  $T$  or  $B$ . We specialize these to panels of type  $T$ . We obtain the following canonical

snakes, representing the  $\Gamma$  and  $\Delta'$  arrays.



Now looking at the even numbered locations in these indexings, starting with  $\gamma_L = \delta'_L$ , Proposition 2 asserts the values  $\nu_1, \dots, \nu_d$  are as advertised in the  $\Delta'$  labeling. Looking at the first odd numbered location, which is the first spot in the bottom row of the episode, Proposition 1 asserts its value to be  $(s - \gamma_L) + \gamma_L - \nu_1 = \mu_1$ ; the second odd numbered location gets the value  $\mu_1 + \nu_1 - \nu_2 = \mu_2$ , and so forth.  $\square$

**Proposition 6** *Let  $\mathfrak{t}$  be a short pattern whose cartoon contains a Class I resonant episode  $\mathcal{E}$  of order  $d$ . Then there exist integers  $s, \mu_1, \nu_1, \mu_2, \nu_2, \dots, \mu_d, \nu_d$  such that  $\mu_i + \nu_i = s$  ( $i = 1, \dots, d$ ), and the portions of in  $\Gamma$  and  $\Delta'$  arrays in  $\mathcal{E}$  are given in the following table.*

$\Gamma_{\mathfrak{t}}$	
$\Delta'_{\mathfrak{t}}$	

The values  $s, \gamma_L$  and  $\gamma_R$  are constant on the type containing the pattern.

**Proof** We define  $s$ ,  $\mu_i$  and  $\nu_i$  to be the quantities that make the  $\Gamma$  array correct. The correctness of the second diagram may be proved using snakes as in Proposition 5. The proof that  $\mu_i + \nu_i = s$  is also similar to Proposition 5.  $\square$

**Proposition 7** Let  $\mathfrak{t}$  be a short pattern whose cartoon contains a Class III resonant episode  $\mathcal{E}$  of order  $d$ . Then there exist integers  $s, \mu_1, \nu_1, \mu_2, \nu_2, \dots, \mu_d, \nu_d$  such that  $\mu_i + \nu_i = s$  ( $i = 1, \dots, d$ ), and the portions of the  $\Gamma$  and  $\Delta'$  arrays in  $\mathcal{E}$  are given in the following table.

$\Gamma_{\mathfrak{t}}$	
$\Delta_{\mathfrak{t}}$	

The values  $s, \gamma_L, \gamma_R$  and  $\xi$  are constant on the type containing the pattern.

**Proof** We define  $s$ ,  $\mu_i$  and  $\nu_i$  to be the quantities that make the  $\Gamma$  array correct. The correctness of the second diagram may be proved using snakes as in Proposition 5. The proof that  $\mu_i + \nu_i = s$  is also similar to Proposition 5.  $\square$

**Proposition 8** Let  $\mathfrak{t}$  be a short pattern whose cartoon contains a Class IV resonant episode  $\mathcal{E}$  of order  $d$ . Then there exist integers  $s, \mu_1, \nu_1, \mu_2, \nu_2, \dots, \mu_d, \nu_d$  such that  $\mu_i + \nu_i = s$  ( $i = 1, \dots, d$ ), and the portions of the  $\Gamma$  and  $\Delta'$  arrays in  $\mathcal{E}$  are given in the following table.

$\Gamma_{\mathfrak{t}}$	
$\Delta_{\mathfrak{t}}$	

The values  $s, \gamma_L, \gamma_R$  and  $\xi$  are constant on the type containing the pattern.

**Proof** We define  $s$ ,  $\mu_i$  and  $\nu_i$  to be the quantities that make the  $\Gamma$  array correct. The correctness of the second diagram may be proved using snakes as in Proposition 5. The proof that  $\mu_i + \nu_i = s$  is also similar to Proposition 5.  $\square$

## 8 Knowability

We refer to Section 3 for discussion of the concept of Knowability.

Let  $\mathfrak{S} = \prod \mathfrak{S}_i$  be a type. Let  $\mathcal{E} = \mathcal{E}_i$  be an episode in the cartoon associated to the short Gelfand-Tsetlin pattern  $\mathfrak{t} \in \mathfrak{S}$ , and let  $a_1, \dots, a_d$  and  $L_1, \dots, L_{d+1}$  be as in Proposition 5, 6, 7 or 8, depending on the class of  $\mathcal{E}$ . We say that  $\mathfrak{t}$  is  $\mathcal{E}$ -maximal if  $a_1 = L_1, \dots, a_d = L_d$ , and  $\mathcal{E}$ -minimal if  $a_1 = L_2, \dots, a_d = L_{d+1}$ . Not every local type  $\mathfrak{S}_i$  (with  $\mathcal{E} = \mathcal{E}_i$ ) contains an  $\mathcal{E}$ -maximal or  $\mathcal{E}$ -minimal element. If it does, then  $\mathfrak{S}_i$  consists of that single local pattern.

**Proposition 9** *If  $\mathfrak{t}_i$  is  $\mathcal{E}_i$ -maximal then  $\mathfrak{t}'_i$  is  $\mathcal{E}_i$ -minimal, and*

$$G_{\Delta}^{\mathcal{E}_i}(\mathfrak{t}'_i) = q^{\gamma_{L_{\mathcal{E}_i}} - \gamma_{R_{\mathcal{E}_i}}} G_{\Gamma}^{\mathcal{E}_i}(\mathfrak{t}_i).$$

**Proof** Let notations be as in Proposition 5, 6, 7 or 8, depending on the class of  $\mathcal{E} = \mathcal{E}_i$ , and in particular  $L = L_{\mathcal{E}_i}$  and  $R = R_{\mathcal{E}_i}$ . Each entry in the  $\mathcal{E}$ -portion of both  $\Gamma_{\mathfrak{t}}$  and  $\Delta_{\mathfrak{t}'}$  is boxed except  $\gamma_L$  and  $\gamma_R$ , if they happen to lie inside  $\mathcal{E}$ , which one or both does unless  $\mathcal{E}$  is of Class II; these are neither boxed nor circled. We have, therefore

$$G_{\Gamma}^{\mathcal{E}_i}(\mathfrak{t}_i) = g(s - \gamma_L) \prod g(\mu_i)g(\nu_i) \times \begin{cases} h(\gamma_L) & \text{Class I or IV} \\ 1 & \text{Class II or III} \end{cases} \times \begin{cases} h(\gamma_R) & \text{Class I or III} \\ 1 & \text{Class II or IV} \end{cases}$$

and  $G_{\Delta}^{\mathcal{E}_i}(\mathfrak{t}'_i)$  is the same, except that  $g(s - \gamma_L)$  is replaced by  $g(s - \gamma_R)$ . By Proposition 3 we may assume as usual that  $n|\gamma_L$  and  $n|\gamma_R$ . It follows that  $q^{\gamma_L - \gamma_R} g(s - \gamma_L) = g(s - \gamma_R)$ , and the statement is proved.  $\square$

**Proposition 10 (Knowability Lemma)** *Let  $\mathcal{E}$  be an episode in the cartoon associated to the short Gelfand-Tsetlin pattern  $\mathfrak{t}$ , and let  $L = L_{\mathcal{E}}$  and  $R = R_{\mathcal{E}}$  as in Tables 2 and 3. Let  $s$ ,  $\mu_i$  and  $\nu_i$  be as in Proposition 5, 6, 7 or 8, depending on the class of  $\mathcal{E}$ . Assume that  $n \nmid s$ . Then either  $G_{\Gamma}(\mathfrak{t}) = G_{\Delta}(\mathfrak{t}') = 0$ , or  $\mathfrak{t}$  is  $\mathcal{E}$ -maximal.*

The term ‘‘Knowability Lemma’’ should be understood as follows. It asserts that one of the following cases applies:

- Maximality:  $\mathfrak{t}$  is  $\mathcal{E}$ -maximal, and  $\mathfrak{S}_i$  consists of the single local pattern. In this case (41) follows from Proposition 9. Thus we may assume that  $n|s$ .
- Knowability:  $n|s$  in which case all the Gauss sums that appear in all the patterns of the resotope appear in knowable combinations –  $g(s)$  by itself or  $g(\mu_i)g(\nu_i)$  where  $\mu_i + \nu_i = s$ . Knowability (as explained in the introduction) *per se* is not important for the proof that Statement C implies Statement B, but the precise statement in Proposition 10, particularly the fact that we may assume that  $n|s$ , *will* be important. Theorems 6 and 7 below validate the term “knowability” by explicitly evaluating the sums that arise when  $n|s$ .
- In all other cases where  $n \nmid s$  we have and  $G_\Gamma(\mathfrak{t}) = G_\Delta(\mathfrak{t}') = 0$  for all patterns so (41) is obvious.

**Proof** We will discuss the cases where  $\mathcal{E}$  is Class II or Class I, leaving the remaining two cases to the reader.

First assume that  $\mathcal{E}$  is Class II. Let notations be as in Proposition 5. By Proposition 3 we may assume that  $n|\gamma_L$  and  $n|\gamma_R$ . We will assume that  $G_\Gamma(\mathfrak{t}) \neq 0$  and show that  $s - \gamma_L$ ,  $\mu_i$  and  $\nu_i$  are all boxed. A similar argument would give the same conclusion assuming  $G_\Delta(\mathfrak{t}') \neq 0$ . Since  $h(x) = 0$  when  $n \nmid x$ , if such  $x$  appears in  $\Gamma_{\mathfrak{t}}$  it is either boxed or circled. In particular  $s - \gamma_L$  is either boxed or circled.

We will argue that  $s - \gamma_L$  is not circled. By the Circling Lemma (Lemma 6),  $\mu_1 = s - \gamma_L$ ,  $\nu_1 = \gamma_L$ , and  $\nu_1$  is also circled. Now  $n \nmid \mu_1 = s - \gamma_L$ , so  $\mu_1$  is either circled or boxed, and it cannot be boxed, because this would imply that  $\nu_1$  is both circled and boxed, which is impossible since  $G_\Gamma(\mathfrak{t}) \neq 0$ . Thus  $\mu_1 = s - \gamma_L$  is circled, and we may repeat the argument, showing that  $s - \gamma_L = \mu_1 = \mu_2 = \dots$  so that  $\nu_1 = \nu_2 = \dots$ , and that all entries are circled. When we reach the end of the top row,  $\mu_d$  is circled, which implies that  $\mu_d = 0$ , and so  $s = \gamma_L$ , which is a contradiction since we assumed that  $n \nmid s$ .

This proves that  $s - \gamma_L$  is boxed. Now we argue by contradiction that the  $\mu_i$  and  $\nu_i$  are also boxed. If not, let  $i \geq 0$  be chosen so that  $\nu_1, \dots, \nu_{i-1}$  are boxed (and therefore, so are  $\mu_1, \dots, \mu_{i-1}$ ) but  $\nu_i$  is not boxed. We note that  $\nu_i$  cannot be circled, because if  $\nu_i$  is circled, then  $\mu_{i-1}$  (or  $s$  if  $i = 0$ ) is both circled and boxed, which is a contradiction. Thus  $\nu_i$  is neither boxed nor circled and so  $n|\nu_i$ . Since  $\nu_i + \mu_i = s$  and  $n \nmid s$ , we have  $n \nmid \mu_i$  and so  $\mu_i$  is either boxed or circled. It cannot be boxed since this would imply that  $\nu_i$  is also boxed, and our assumption is that it is not. Thus  $\mu_i$  is circled. By the Circling Lemma,  $\mu_i = \mu_{i+1}$ , and so  $n \nmid \mu_{i+1}$  which is thus either boxed or circled. It cannot be circled, since if it is, then  $\nu_{i+1}$  is both circled (since  $\mu_i$  is circled) and boxed (since  $\mu_{i+1}$  is boxed), and we know that if a bottom

row entry is both boxed and circled, then  $\mathfrak{t}$  is not strict and  $G_\Gamma(\mathfrak{t}) = 0$ , which is a contradiction. Thus  $\mu_{i+1}$  is circled. Repeating this argument,  $\mu_i = \mu_{i+1} = \dots$  are all circled, and when we get to the end,  $\mu_d$  is circled, so by the Circling Lemma,  $\mu_i = \mu_d = \gamma_R$ , which is a contradiction since  $\gamma_R$  is divisible by  $n$ , but  $\mu_i$  is not. This contradiction shows that  $s - \gamma_L$  and the  $\mu_i, \nu_i$  are all boxed, and it follows from the definitions that  $\mathfrak{t}$  is  $\mathcal{E}$ -maximal.

We now discuss the variant of this argument for the case that  $\mathcal{E}$  is of Class I, leaving the two other cases to the reader. Let notations be as in Proposition 6. Again we assume that  $G_\Gamma(\mathfrak{t}) \neq 0$ , so whenever  $x$  appears in  $\Gamma_{\mathfrak{t}}$  with  $n \nmid x$  it is either boxed or circled. Due to its location in the cartoon, there is no way that  $s - \gamma_L$  can be circled, so it is boxed.

Now we argue by contradiction that  $\nu_1, \dots, \nu_d$  and hence  $\mu_1, \dots, \mu_d$  are all boxed. If not, let  $i \geq 0$  be chosen so that  $\nu_1, \dots, \nu_{i-1}$  are boxed (and therefore, so are  $\mu_1, \dots, \mu_{i-1}$ ) but  $\nu_i$  is not boxed. The same argument as in the Class II case shows that  $n \mid \nu_i$  so  $n \nmid \mu_i$  and that  $\mu_i$  is circled, and moreover that  $\mu_i = \mu_{i+1} = \dots = \mu_d$  and that these are all circled. But now this is a contradiction since due to its location in the cartoon,  $\mu_d$  cannot be circled.  $\square$

## 9 The Reduction to Statement D

We now switch to the language of resotopes, as defined in Section 3. We remind the reader that we may assume  $\gamma_{L\mathcal{E}}$  and  $\gamma_{R\mathcal{E}}$  are multiples of  $n$  for every totally resonant episode.

**Proposition 11** *Statement D is equivalent to Statement C. Moreover, Statement D is true if  $n \nmid s$ .*

**Proof** The case of a totally resonant short Gelfand-Tsetlin pattern  $\mathfrak{t}$  is a special case of Proposition 5, and the point is that  $\Gamma_{\mathfrak{t}}$  is a  $\Gamma$ -accordion  $\mathfrak{a}$ , and Proposition 5 shows that  $\Delta_{\mathfrak{t}}$  is the  $\Delta$ -accordion  $\mathfrak{a}'$ . In this case  $\gamma_L = \gamma_R = 0$ . Moreover as  $\mathfrak{t}$  runs through its totally resonant prototype,  $\mathfrak{a}$  runs through the  $\Gamma$ -resotope  $\mathcal{A}_s(c_0, \dots, c_d)$  with  $c_i = L_i - L_{i+1}$ , so Statement D boils down to Statement C. The fact that Statement D is true when  $n \nmid s$  follows from the Knowability Lemma and Proposition 9.  $\square$

We turn next to the proof that Statement D implies Statement B. What we will show is that for each of the four types of resonant episodes, Statement D implies (41); then Statement B will follow from Proposition 4. We fix an episode  $\mathcal{E} = \mathcal{E}_i$ , and will denote  $L = L_i, R = R_i$ . By Proposition 3 we may assume that  $n \mid \gamma_L$  and  $n \mid \gamma_R$ . Moreover by the Knowability Lemma (Proposition 10) we may assume  $n \mid s$ ,

where  $s$  and other notations are as in Proposition 5, 6, 7 or 8, depending on the class of  $\mathcal{E}$ .

**Proposition 12** *Let  $\mathcal{E}$  be a Class II episode, and let notations be as in Proposition 5. If*

$$\mathbf{a} = \left\{ \begin{array}{ccccccc} \hat{s} & \hat{\mu}_1 & & \cdots & & \hat{\mu}_d & \\ & \hat{\nu}_1 & & \cdots & & \hat{\nu}_d & \end{array} \right\}$$

where  $s = \hat{s} + \gamma_R + \gamma_L$ ,  $\mu_i = \hat{\mu}_i + \gamma_R$  and  $\nu_i = \hat{\nu}_i + \gamma_L$ , then  $\mathbf{a}$  lies in the resotope  $\mathcal{A} = \mathcal{A}_{\hat{s}}(c_0, \dots, c_d)$  with  $c_i = L_i - L_{i+1}$ ; let  $\sigma$  denote the signature of  $\mathbf{a}$  in  $\mathcal{A}$ . Then  $\mathbf{t} \mapsto \mathbf{a}_\sigma$  induces a bijection from the local type  $\mathfrak{S}_i$  to  $\mathcal{A}$ . Assume furthermore that  $n|\gamma_L$  and  $n|\gamma_R$ . Then

$$q^{\gamma_L} G_\Gamma^\mathcal{E}(\mathbf{t}) = q^{(d+1)(\gamma_R+\gamma_L)} \mathcal{G}_\Gamma(\mathbf{a}_\sigma), \quad q^{\gamma_R} G_\Delta^\mathcal{E}(\mathbf{t}') = q^{(d+1)(\gamma_R+\gamma_L)} \mathcal{G}_\Delta(\mathbf{a}'_\sigma). \quad (43)$$

**Proof** With notations as in Proposition 5, the inequalities  $L_i \geq a_i \geq L_{i+1}$  that  $a_i$  must satisfy can be written (with  $\hat{\mu}_0 = \hat{s}$ ):

$$L_i - L_{i+1} \geq \hat{\mu}_{i-1} - \hat{\mu}_i \geq 0,$$

which are the same as the conditions that  $\mathbf{a}_\sigma$  lies in  $\mathcal{A} = \mathcal{A}_s(c_0, \dots, c_d)$ , with  $c_i = L_i - L_{i+1}$ . Each entry in  $\mathbf{a}_\sigma$  is boxed or circled if and only if the corresponding entry in  $\Gamma_{\mathbf{t}}$  is, and similarly, every entry in  $\mathbf{a}'_\sigma$  is boxed or circled if and only if the corresponding entry in the (left-to-right) mirror image of  $\Delta_{\mathbf{t}'}$  is. Using the assumption that  $n|\gamma_R$  and  $n|\gamma_L$  and Proposition 1 we can pull a factor of  $q^{\gamma_R}$  from the factor of  $G_\Gamma^\mathcal{E}(\mathbf{t})$  corresponding to  $s - \gamma_L = \hat{s} + \gamma_R$ , which is

$$\begin{cases} g(\hat{s} + \gamma_R) & \text{if } s - \gamma_L \text{ is boxed;} \\ q^{\hat{s} + \gamma_R} & \text{if } s - \gamma_L \text{ is circled;} \\ h(\hat{s} + \gamma_R) & \text{otherwise,} \end{cases}$$

leaving just the corresponding contribution in  $\mathcal{G}_\Gamma(\mathbf{a}_\sigma)$ ; and similarly we may pull out  $d$  factors of  $q^{\gamma_R}$  from the contributions of  $\mu_i = \hat{\mu}_i + \gamma_R$ , and  $d$  factors of  $q^{\gamma_L}$  from the contributions of  $\nu_i = \hat{\nu}_i + \gamma_L$ . What remains is just  $\mathcal{G}_\Gamma(\mathbf{a}_\sigma)$ . This gives the first identity in (43), and the second one is proved similarly.  $\square$

**Corollary.** *Statement D implies (41) for Class II episodes.*

Although this reduction was straightforward for Class I, each of the remaining classes involves some nuances. In every case we will argue by comparing  $q^{\gamma_L} G_\Gamma(\mathbf{t})$  to



We will compare  $G_\Gamma(\mathbf{t})$  with  $\mathcal{G}_\Gamma(\mathbf{a}_\sigma)$ , where  $\mathbf{a}$  is the accordion (45) derived from the pattern in (44), and  $\sigma$  is its signature. Thus we move  $L_0$  and  $L_{d+1}$  to the top row, which does not affect the inequalities that the  $a_i$  satisfy, and discard the rest of the pattern to obtain the totally resonant pattern (44), then compute its accordion. Otherwise, let  $\Gamma_{\mathbf{t}}$ , and  $\Delta_{\nu}$  be as in Proposition 6, and let  $s$ ,  $\mu_i$  and  $\nu_i$  be as defined there.

**Proposition 13** *Assume  $\mathcal{E}$  is a Class I episode and that  $n|s, \gamma_L, \gamma_R$ . As  $\mathbf{t}$  runs through its local type,  $\mathbf{a}_\sigma$  runs through  $\mathcal{A}_{\mathbf{t}}(c_0, \dots, c_d)$  with  $c_i = L_i - L_{i+1}$ , and*

$$\begin{aligned} q^{\gamma_L} G_\Gamma^{\mathcal{E}}(\mathbf{t}) &= h(\gamma_L) h(\gamma_R) q^{(d+1)(2s-\gamma_L-\gamma_R)} \mathcal{G}_\Gamma(\mathbf{a}_\sigma), \\ q^{\gamma_R} G_\Delta^{\mathcal{E}}(\mathbf{t}') &= h(\gamma_L) h(\gamma_R) q^{(d+1)(2s-\gamma_L-\gamma_R)} \mathcal{G}_\Delta(\mathbf{a}'_\sigma). \end{aligned}$$

**Proof** Using (46) and (44), we have

$$t = \sum_{j=0}^d (a_j - L_{j+1}) = \gamma_R - (s - \gamma_L), \quad \psi_i = \sum_{j=i}^d (a_j - L_{j+1}) = \gamma_R - \nu_i,$$

and  $\phi_i + \psi_i = t$ . If  $\gamma_L$  is circled, we will move the circle to  $s - \gamma_L$ . To justify the use of the Moving Lemma (Lemma 8) we check that  $\gamma_L$  and  $s - \gamma_L$  are both positive and congruent to zero modulo  $n$ . Positivity of  $\gamma_L$  follows since  $\gamma_L \geq \mu_d$ , and  $\mu_d > 0$  since if  $\mu_d = 0$  then it is circled, which it cannot be due to its location in the cartoon. To see that  $s - \gamma_L > 0$ , if it is zero then both  $s - \gamma_L$  and  $\gamma_L$  are circled, which implies that  $L_1 = a_1 = L_2$ , but  $L_1 > L_2$ . Both  $s - \gamma_L$  and  $\gamma_L$  are multiples of  $n$  by assumption.

If  $\gamma_R$  is circled, we will move the circle to  $\mu_d$ . To see that this is justified, we must check that  $\gamma_R$  and  $\mu_d$  are positive and multiples of  $n$ . We are assuming  $n|\gamma_R$ , and it is positive since  $\gamma_R \geq s - \gamma_L$  which cannot be zero; if it were, it would be circled, which it cannot be due to its position in the cartoon. Also by the Circling Lemma, since  $\gamma_R$  is circled it equals  $\nu_d$ ; thus  $\mu_d = s - \nu_d = s - \gamma_R \equiv 0$  modulo  $n$ . And  $\mu_d$  cannot be zero since it is not circled, due to its location in the cartoon.

With these circling modifications,  $\gamma_L$  and  $\gamma_R$  are neither circled nor boxed, hence produce factors in  $G_\Gamma^{\mathcal{E}}(\mathbf{t})$  of  $h(\gamma_L)$  and  $h(\gamma_R)$ . The remaining factors in  $G_\Gamma^{\mathcal{E}}(\mathbf{t})$  can be handled as follows. Let  $F(x) = q^x$  if  $x$  is a boxed entry in  $\Gamma_{\mathbf{t}}$  or  $\mathbf{a}_\sigma$ ,  $g(x)$  if it is circled, and  $h(x)$  if it is neither boxed nor circled. We have

$$\begin{aligned} q^{\gamma_L} F(s - \gamma_L) &= q^{2s-\gamma_L-\gamma_R} F(t), \\ F(\mu_i) &= q^{s-\gamma_R} F(\psi_i), \\ F(\nu_i) &= q^{s-\gamma_L} F(\phi_i), \end{aligned}$$

and multiplying these identities together gives the stated identity for  $q^{\gamma_L} G_\Gamma^\mathcal{E}(\mathbf{t})$ . (There are two entries  $h(\gamma_L)$  and  $h(\gamma_R)$  that have to be taken out.) The  $\Delta'$  array is handled similarly.  $\square$

### Class III episodes

Now we assume that the  $\mathcal{E}$ -portion of  $\mathbf{t}$  has the form:

$$\begin{array}{ccccccc}
 & L_0 & & L_1 & & L_2 & \cdots & L_d & & \circ & & \circ \\
 & \diagdown & & \diagup & \diagdown & \diagup & & \diagdown & & \diagup & & \diagdown \\
 \circ & & a_0 & & a_1 & & \cdots & & a_d & & \circ \\
 & \diagup & & \diagdown & \diagup & \diagdown & & \diagup & & \diagdown & & \diagup \\
 & L_1 & & L_2 & & \cdots & L_r & & L_{d+1} & & \circ
 \end{array} \tag{47}$$

**Proposition 14** *Assume  $\mathcal{E}$  is a Class III episode and that  $n|s, \gamma_L, \gamma_R$ . If  $a_0 = L_1, a_1 = L_2, \dots, a_d = L_{d+1}$  then the local type consists of a single pattern  $\mathbf{t}$ , for which (41) is satisfied. Assume that this is not the case. Then as  $\mathbf{t}$  runs through its local type,  $\mathbf{a}_\sigma$  runs through  $\mathcal{A}_t(c_0, \dots, c_d)$  with  $c_i = L_i - L_{i+1}$ , and*

$$q^{\gamma_L} G_\Gamma^\mathcal{E}(\mathbf{t}) = q^{(d+1)(s+\gamma_L-\gamma_R)} h(\gamma_R) \mathcal{G}_\Gamma(\mathbf{a}_\sigma), \quad q^{\gamma_R} G_\Delta^\mathcal{E}(\mathbf{t}') = q^{(d+1)(s+\gamma_L-\gamma_R)} h(\gamma_R) \mathcal{G}_\Delta(\mathbf{a}'_\sigma).$$

**Proof** If  $a_0 = L_1, a_1 = L_2, \dots, a_d = L_{d+1}$  then the local type consists of a single element  $\mathbf{t}$ . We will handle this case separately. For this  $\mathbf{t}$  it is easy to see that all entries except  $\mu_d$  are circled in  $\Gamma_{\mathbf{t}}$ , while in  $\Delta_{\mathbf{t}}$  all entries except  $s - \gamma_R$  are circled. But by the Circling Lemma  $s - \gamma_L = \mu_1 = \dots = \mu_d$  and  $\mu_d > 0$  since it cannot be circled due to its location in the cartoon. Thus we may move the circle from  $s - \gamma_L$  to  $\mu_d$  and then compare  $G_\Gamma^\mathcal{E}(\mathbf{t})$  and  $G_\Delta^\mathcal{E}(\mathbf{t}')$  to see directly that (41) is true.

We exclude this case and assume that at least one of the inequalities  $a_i \geq L_{i+1}$  is strict. Using (45) and (47) we have  $t = \gamma_R - \gamma_L, \phi_i = \gamma_R - \gamma_L, \psi_i = \nu_i - \gamma_L$  and  $\psi_i = \mu_i + \gamma_R - s$ , where  $\gamma_R, \gamma_L, s, \mu_i$  and  $\nu_i$  are as in Proposition 7, and  $\sigma$  is the signature of  $\mathbf{a}$  in  $\mathcal{A}$ . If  $\gamma_L$  is circled then we move the circle from  $\gamma_L$  to  $\mu_d$  in  $\Gamma_{\mathbf{t}}$ . This is justified as in the Class I case, except that the justification we gave there for the claim that  $\gamma_R > 0$  is no longer valid. It follows now from our assumption that one of the inequalities  $a_i \geq L_{i+1}$  is strict. After moving the circle from  $\gamma_L$  to  $\mu_d$  in  $\Gamma_{\mathbf{t}}$ , each factor  $s - \gamma_L, \mu_i, \nu_i$  is circled or boxed in the (circling-modified)  $\Gamma_{\mathbf{t}}$  if and only if the corresponding factor  $t, \psi_i$  or  $\phi_i$  is circled or boxed in  $\mathbf{a}_\sigma$ . Moreover  $s + \gamma_L - \gamma_R \equiv 0$  modulo  $n$  so we can pull out a factor of  $q^{s+\gamma_L-\gamma_R}$  from the contributions of  $s - \gamma_L$  and each pair  $\mu_i, \nu_i$ , to  $q^{\gamma_L} G_\Gamma^\mathcal{E}(\mathbf{t})$ , and what remains is  $h(\gamma_R) \mathcal{G}_\Gamma(\mathbf{a}_\sigma)$ . A similar treatment gives the other identity.  $\square$



## 10 Statement E implies Statement D

We fix a nodal signature. Let  $B(\eta) = \{i | \eta_i = \square\}$ . Let  $\mathcal{CP}_\eta(c_0, \dots, c_d) \in \mathfrak{Z}_\Gamma$  be the following “cut and paste” virtual resotope

$$\mathcal{CP}_\eta(c_0, \dots, c_d) = \sum_{T \subseteq B(\eta)} (-1)^{|T|} \mathcal{A}_s(c_0^T, \dots, c_d^T), \quad (49)$$

where

$$c_i^T = \begin{cases} c_i & \text{if } i \in T, \\ \infty & \text{if } i \notin T. \end{cases}$$

We recall that  $\text{CP}_\eta(c_0, \dots, c_d)$  is the set of  $\Gamma$ -accordions

$$\mathbf{a} = \left\{ \begin{array}{ccccccc} s & \mu_1 & & \cdots & & \mu_d & \\ & \nu_1 & & \cdots & & \nu_d & \end{array} \right\}$$

that satisfy the inequalities (25), with the convention that  $\mu_0 = s$  and  $\mu_{d+1} = 0$ . Geometrically, this set is a simplex, and we will show that it is the support of  $\mathcal{CP}_\eta(c_0, \dots, c_d)$ , though the latter virtual resotope is a superposition of resotopes whose supports include elements that are outside of  $\text{CP}_\eta(c_0, \dots, c_d)$ ; it will be shown that the alternating sum causes such terms to cancel.

Finally, if  $\mathbf{a} \in \text{CP}_\eta(c_0, \dots, c_d)$  let  $\theta(\mathbf{a}, \eta)$  be the signature obtained from  $\eta$  by changing  $\eta_i$  to  $*$  when the inequality

$$\mu_i - \mu_{i+1} \geq \begin{cases} c_i & \text{if } \eta_i = \square, \\ 0 & \text{if } \eta_i = \circ, \end{cases}$$

is strict. Note that these are the inequalities defining  $\mathbf{a} \in \text{CP}_\eta(c_0, \dots, c_d)$ . Strictly speaking  $\mathbf{a}$  and  $\eta$  do not quite determine  $\theta(\mathbf{a}, \eta)$  because it also depends on the  $c_i$ . We omit these data since they are fixed, while  $\mathbf{a}$  and  $\eta$  will vary.

**Proposition 16** *The support of  $\mathcal{CP}_\eta(c_0, \dots, c_d)$  is the simplex  $\text{CP}_\eta(c_0, \dots, c_d)$ . Suppose that  $\mathbf{a} \in \text{CP}_\eta(c_0, \dots, c_d)$ . If  $\tau$  is any signature, then the coefficient of  $\mathbf{a}_\tau$  in  $\mathcal{CP}_\eta(c_0, \dots, c_d)$  is zero unless  $\tau$  is obtained from  $\theta(\mathbf{a}, \eta)$  by changing some  $\square$ 's to  $*$ . If it is so obtained, the coefficient is  $(-1)^\varepsilon$ , where  $\varepsilon$  is the number of  $\square$ 's in  $\tau$ .*

**Proof** Suppose the  $\Gamma$ -accordion  $\mathbf{a}$  does not satisfy (25). We will show that it does not appear in the support of  $\mathcal{CP}_\eta(c_0, \dots, c_d)$ . By assumption  $\mu_i - \mu_{i+1} < c_i$  for some  $i \in B(\eta)$ . We group the subsets of  $B(\eta)$  into pairs  $T, T'$  where  $T = T' \cup \{i\}$ . It is clear that  $\mathbf{a}$  occurs in  $\mathcal{A}_s(c_0^T, \dots, c_d^T)$  if and only if it occurs in  $\mathcal{A}_s(c_0^{T'}, \dots, c_d^{T'})$ , and

with the same signature. Since these have opposite signs, their contributions cancel. This proves that the support of  $\mathcal{CP}_\eta(c_0, \dots, c_d)$  is contained in the simplex  $\mathcal{C}$ . The opposite inclusion will be clear from the precise description of the coefficients, which is our next step to prove.

We note that  $\theta(\mathbf{a}, \eta) = \theta_0 \cdots \theta_d$  where

$$\theta_i = \begin{cases} \square & \text{if } \mu_i - \mu_{i+1} = c_i, \\ \circ & \text{if } \mu_i - \mu_{i+1} = 0, \\ * & \text{otherwise.} \end{cases}$$

We emphasize that if  $\theta_i = \square$  then  $i \in B(\eta)$ , while if  $\theta_i = \circ$  then  $i \notin B(\eta)$ . (The case  $\theta_i = *$  can arise whether  $i \in B(\eta)$ .)

Suppose that  $\mathbf{a} \in \mathcal{CP}_\eta(c_0, \dots, c_d)$ . In order for  $\mathbf{a}_\tau$  to have a nonzero coefficient, it must appear as the coefficient of  $\mathbf{a}$  in  $\mathcal{A}_s(c_0^T, \dots, c_d^T)$  for some subset  $T$  of  $B(\eta)$ . We will prove that if  $\tau$  is the signature of  $\mathbf{a}$  in this resohedron we have

$$\tau_i = \begin{cases} \circ & \text{if } \mu_i - \mu_{i+1} = 0, \text{ in which case } i \notin B(\eta); \\ \square & \text{if } i \in T; \\ * & \text{otherwise.} \end{cases} \quad (50)$$

First, if  $i \in T$  then  $c_i^T = c_i$  so  $\mu_i - \mu_{i+1} \leq c_i$ , while for we have already stipulated (by assuming  $\mathbf{a} \in \mathcal{CP}_\eta(c_0, \dots, c_d)$ ) that  $\mu_i - \mu_{i+1} \geq c_i$ . Therefore  $\mu_i - \mu_{i+1} = c_i$  when  $i \in T$ , and so  $\tau_i = \square$  when  $i \in T$ . And if  $i \notin T$ , the signature of  $\tau$  is definitely not  $\square$  since  $c_i^T = \infty$ ; if  $i \in B(\eta) - T$  it also cannot be  $\circ$  since  $\mu_i - \mu_{i+1} \geq c_i > 0$ . This proves (50).

It is clear from (50) that  $\tau$  is obtained from  $\theta(\mathbf{a})$  by changing some  $\square$ 's to  $*$ 's, and which ones are changed determines  $T$ . This point is important since it shows that (unlike the case where  $\mathbf{a} \notin \mathcal{CP}_\eta(c_0, \dots, c_d)$ ) a given  $\mathbf{a}_\tau$  can only appear in only one term in (49), so there cannot be any cancellation. If  $\tau$  is obtained from  $\theta(\mathbf{a})$  by changing some  $\square$ 's to  $*$ 's then it does appear in  $\mathcal{A}_s(c_0^T, \dots, c_d^T)$  for a unique  $T$  and so  $\mathbf{a}_\tau$  appears in  $\mathcal{CP}_\eta(c_0, \dots, c_d)$  with a nonzero coefficient. The sign with which it appears is  $(-1)^{|T|}$ , and  $T$  we have noted is the set of  $i$  for which  $\tau_i = \square$ .  $\square$

**Theorem 5** *Statement E implies Statement D.*

**Proof** Let  $\mathbf{a} \in \mathcal{CP}_\eta(c_0, \dots, c_d)$  and let  $\sigma = \theta(\mathbf{a}, \eta)$ . What we must show is that (26) implies (23). We extend the function  $G_\Gamma$  from the set of decorated  $\Gamma$ -accordions to the free abelian group  $\mathfrak{Z}_\Gamma$  by linearity. Also the involution  $\mathbf{a}_\eta \mapsto \mathbf{a}'_\eta$  on decorated accordions induces an isomorphism  $\mathfrak{Z}_\Gamma \rightarrow \mathfrak{Z}_\Delta$  that we will denote  $\mathcal{A} \mapsto \mathcal{A}'$ .

Then (23) can be written  $\mathcal{G}_\Gamma(\mathcal{A}) = \mathcal{G}_\Delta(\mathcal{A}')$ . By the principle of inclusion-exclusion (Stanley [28], page 64), we have

$$\mathcal{A}_s(c_0, \dots, c_d) = \sum_{T \subseteq B(\eta)} \mathcal{CP}_{\eta^T}(c_0^T, \dots, c_d^T),$$

Where if  $T$  is a subset of  $B(\eta)$  then  $\eta^T$  is the signature obtained by changing  $\eta_i$  from  $\square$  to  $\circ$  for all  $i \in T$ . This means that if we show  $\mathcal{G}_\Gamma(\mathcal{C}) = \mathcal{G}_\Delta(\mathcal{C}')$  when  $\mathcal{C} = \mathcal{CP}_\eta(c_0, \dots, c_d)$  then (23) will follow. The left-hand side in this identity is a sum of  $\mathcal{G}_\Gamma(\mathbf{a}_\tau)$  with  $\mathbf{a}$  in  $\mathcal{CP}_\eta(c_0, \dots, c_d)$ , and the coefficient of  $\mathbf{a}_\tau$  in this sum is the same as its coefficient in  $\Lambda_\Gamma(\mathbf{a}, \sigma)$  by Proposition 16.  $\square$

## 11 Evaluation of $\Lambda_\Gamma$ and $\Lambda_\Delta$ , and Statement G

Let  $\eta$  be a nodal signature, and let  $\sigma$  be a subsignature. Let

$$\mathbf{a} = \left\{ \begin{array}{cccccc} s & \alpha_1 & \alpha_2 & \cdots & \alpha_d \\ \beta_1 & \beta_2 & \cdots & \beta_d \end{array} \right\}$$

be an accordion belonging to the open facet  $\mathcal{S}_\sigma$  of  $\mathcal{CP}_\eta(c_0, \dots, c_d)$ . Assuming that  $n|s$  we will evaluate  $\Lambda_\Gamma(\mathbf{a}, \sigma)$ .

We will denote

$$V(a, b) = (q-1)^a q^{(d+1)s-b}, \quad V(a) = V(a, a).$$

Let

$$\begin{aligned} \varepsilon_\Gamma(\sigma) = \varepsilon_\Gamma &= \begin{cases} 1 & \text{if } \sigma_0 = \square, \\ 0 & \text{otherwise,} \end{cases} \\ \mathcal{K}_\Gamma(\sigma) = \mathcal{K}_\Gamma &= \{i | 1 \leq i \leq d, \sigma_i = \square, \sigma_{i-1} \neq \circ\}, & k_\Gamma &= |\mathcal{K}_\Gamma|, \\ \mathcal{N}_\Gamma(\sigma) = \mathcal{N}_\Gamma &= \{i | 1 \leq i \leq d, \sigma_i = \square, \sigma_{i-1} = \circ\}, & n_\Gamma &= |\mathcal{N}_\Gamma|, \end{aligned}$$

and

$$\mathcal{C}_\Gamma(\sigma) = \mathcal{C}_\Gamma = \{i | 1 \leq i \leq d, \sigma_0, \sigma_1, \dots, \sigma_{i-1} \text{ not all } \circ \text{ and either } i \in \mathcal{N}_\Gamma \text{ or } \sigma_i = *\}. \quad (51)$$

Let  $c_\Gamma = |\mathcal{C}_\Gamma|$ , and let  $t_\Gamma$  be the number of  $i$  with  $1 \leq i \leq d$  and  $\sigma_i \neq *$ . Given a set of indices  $\Sigma = \{i_1, \dots, i_k\} \subseteq \{1, 2, \dots, d\}$ , let

$$\delta_n(i_1, \dots, i_k) = \delta_n(\Sigma) = \delta_n(\Sigma; \mathbf{a}) = \begin{cases} 1 & \text{if } n \text{ divides } \alpha_{i_1}, \dots, \alpha_{i_k}, \\ 0 & \text{otherwise.} \end{cases} \quad (52)$$

Let

$$\chi_\Gamma(\mathbf{a}, \sigma) = \chi_\Gamma = \prod_{i \in \mathcal{C}_\Gamma(\sigma)} \delta_n(i).$$

Finally, let

$$a_\Gamma(\sigma) = a_\Gamma = 2(d - t_\Gamma + n_\Gamma) + \left\{ \begin{array}{ll} -1 & \text{if } \sigma_0 = \circ \\ 0 & \text{if } \sigma_0 = \square \\ 1 & \text{if } \sigma_0 = * \end{array} \right\} + \left\{ \begin{array}{ll} 1 & \text{if } \sigma_d = \circ \\ 0 & \text{if } \sigma_d \neq \circ \end{array} \right\}.$$

**Proposition 17** *Assume that  $n|s$ . Given a  $\Gamma$ -accordion*

$$\mathbf{a} = \left\{ \begin{array}{cccccc} s & \alpha_1 & \alpha_2 & \cdots & \alpha_d \\ \beta_1 & \beta_2 & \cdots & \beta_d \end{array} \right\}$$

*and an associated signature  $\sigma \subseteq \sigma_{\overline{\alpha}}$  not containing the substring  $\circ, \square$ , then*

$$\mathcal{G}_\Gamma(\mathbf{a}, \sigma) = (-1)^{\varepsilon_\Gamma} \chi_\Gamma \cdot V(a_\Gamma, a_\Gamma + d_\Gamma), \quad (53)$$

where

$$d_\Gamma = \left( \sum_{\substack{1 \leq i \leq d \\ \sigma_i = \square}} (1 + \delta_n(i)) \right) + \left\{ \begin{array}{ll} 1 & \text{if } \sigma_0 = \square \\ 0 & \text{if } \sigma_0 \neq \square \end{array} \right\}.$$

Recall that any subsignature  $\sigma$  containing the string  $\circ \square$  has  $\mathcal{G}_\Gamma(\mathbf{a}, \sigma) = 0$ . We will abuse notation and rewrite the definition of  $\mathcal{G}_\Gamma(\mathbf{a}, \sigma)$  as

$$\mathcal{G}_\Gamma(\mathbf{a}, \sigma) = \mathcal{G}_\Gamma(\mathbf{a}_\sigma) = \prod_{x \in \mathbf{a}} f_\sigma(x),$$

where

$$f_\sigma(x) = \begin{cases} g(x) & \text{if } x \text{ is boxed in } \mathbf{a}_\sigma \text{ (but not circled),} \\ q^x & \text{if } x \text{ is circled (but not boxed),} \\ h(x) & \text{if } x \text{ is neither boxed nor circled,} \\ 0 & \text{if } x \text{ is both boxed and circled.} \end{cases}$$

This is an abuse of notation, since  $f_\sigma$  is not a function; it depends not only on the numerical value  $x$  but also its location in the decorated accordion  $\mathbf{a}_\sigma$ . However this should cause no confusion.

**Proof** Using the signature  $\sigma$  to determine the rules for boxing and circling in  $\mathbf{a}$  we see that if  $\sigma_0 = \square$ , then  $f_\sigma(s) = g(s) = (-1) \cdot q^{s-1}$ . The  $(-1)$  here accounts for the  $(-1)^{\varepsilon_\Gamma}$  in (53). If  $\sigma_i = \square$  for  $i > 0$ , we have

$$f_\sigma(\alpha_i)f_\sigma(\beta_i) = g(\alpha_i)g(\beta_i) = \begin{cases} q^{s-1} & \text{if } n \nmid \alpha_i, \\ q^{s-2} & \text{if } n \nmid \alpha_i. \end{cases}$$

If  $\sigma_0 = \circ$ , then  $s = \alpha_1$ ,  $\beta_1 = 0$ , and  $f_\sigma(s) = q^s$ . If  $\sigma_i = \circ$ ,  $0 < i < d$ , then  $\alpha_i = \alpha_{i+1}$  and  $\beta_i = \beta_{i+1}$  so that while the circling in the accordion strictly speaking occurs at  $\alpha_i$  and  $\beta_{i+1}$ , we may equivalently consider it to occur at  $\alpha_i$  and  $\beta_i$  for bookkeeping purposes and

$$f_\sigma(\alpha_i)f_\sigma(\beta_{i+1}) = f_\sigma(\alpha_i)f_\sigma(\beta_i) = q^s.$$

And if  $\sigma_d = \circ$ , then  $\alpha_d = 0$  and  $\beta_d = s$ , so that

$$f_\sigma(\alpha_d)f_\sigma(\beta_d) = h(s) = (q-1)q^{s-1}.$$

Finally if  $\sigma_0 = *$ , then  $f_\sigma(s) = (q-1)q^{s-1}$ . If  $\sigma_i = *$ ,  $1 \leq i \leq d$  then

$$f_\sigma(\alpha_i)f_\sigma(\beta_i) = h(\alpha_i)h(\beta_i) = \begin{cases} (q-1)^2q^{s-2} & \text{if } n \nmid \alpha_i, \\ 0 & \text{if } n \nmid \alpha_i. \end{cases}$$

Now note that the assumption that  $\sigma$  does not contain the string  $\circ, \square$  implies that  $n_\Gamma = 0$ , simplifying the definitions of  $\chi_\Gamma$  and  $a_\Gamma$  above. The case of  $\sigma_i = \square$  is seen to account for the  $d_\Gamma$  defined above, the  $\sigma_i = *$  account for both the  $\chi_\Gamma$  and the  $a_\Gamma$ . However, one does need to count somewhat carefully at the ends of the accordion according to the above cases. In particular, we see that  $\sigma_0 = \circ$  implies  $\alpha_1 = s$ , so that if  $j$  is the first index with  $\sigma_j \neq \circ$ , then  $\sigma_j = *$  by assumption. But then  $\alpha_j = s$  and the divisibility condition  $n|\alpha_j$  is automatic, hence redundant and omitted from the definition.  $\square$

**Lemma 9** *Given a signature  $\sigma$  which does not contain the sequence  $\circ\square$  and  $d_\Gamma(\sigma)$  as defined in Proposition 17, we may write  $d_\Gamma = k_\Gamma + \varepsilon_\Gamma + \sum_{i \in \mathcal{K}_\Gamma(\sigma)} \delta_n(i)$ . Then for any  $m$  with  $0 \leq m < d_\Gamma$*

$$\begin{aligned} \binom{d_\Gamma}{m} &= k_\Gamma + \binom{\varepsilon_\Gamma}{m} + \sum_{i \in \mathcal{K}_\Gamma(\sigma)} \delta_n(i) \binom{k_\Gamma + \varepsilon_\Gamma}{m-1} + \\ &\quad \cdots + \sum_{\{i_1, \dots, i_l\} \subseteq \mathcal{K}_\Gamma(\sigma)} \delta_n(i_1, \dots, i_l) \binom{k_\Gamma + \varepsilon_\Gamma}{m-l} + \cdots \end{aligned}$$

where we understand each of the binomial coefficients to be 0 if the lower entry is either negative or larger than the upper entry.

**Proof** The result follows from repeated application of the identity

$$\binom{c + \delta_n(i)}{m} = \binom{c}{m} + \delta_n(i) \binom{c}{m-1},$$

valid for any constants  $c$  and  $m$  and index  $i$ , While  $d_\Gamma$  contains divisibility conditions and hence depends on  $\mathbf{a}$ ,  $k_\Gamma + \varepsilon_\Gamma$  is an absolute constant depending only on the signature  $\sigma$ .  $\square$

**Theorem 6** Fix a nodal signature  $\eta$ , and assume that  $n|s$ . Given an accordion  $\mathbf{a} \in \mathcal{S}_\sigma$  with subsignature  $\sigma = (\sigma_0, \sigma_1, \dots, \sigma_d) \subset \eta$ ,

$$\Lambda_\Gamma(\mathbf{a}, \sigma) = (-1)^{n_\Gamma + \varepsilon_\Gamma} \chi_\Gamma \sum_{x=0}^{k_\Gamma} \sum_{\substack{\Sigma \subseteq \mathcal{K}_\Gamma(\sigma) \\ |\Sigma| = x}} \delta_n(\Sigma) (-1)^x V(a_\Gamma + x, a_\Gamma + k_\Gamma), \quad (54)$$

where the inner sum ranges over all possible subsets of cardinality  $x$  in  $\mathcal{K}_\Gamma$ .

Before giving the proof, let's do an example. Let

$$\sigma = (\square, *, \circ, \square, *, \square, *) \subseteq \eta = (\square, \circ, \circ, \square, \square, \square, \circ)$$

then one can read off the following data from the signature:

$$n_\Gamma = 1, \quad \varepsilon_\Gamma = 1, \quad \chi_\Gamma = \delta_n(1, 3, 4, 6), \quad t_\Gamma = 3, \quad \mathcal{K}_\Gamma = \{5\}, \quad k_\Gamma = 1, \quad a_\Gamma = 8$$

so

$$\Lambda_\Gamma(\sigma) = \delta_n(1, 3, 4, 6) (V(8, 9) - \delta_n(5)V(9, 9))$$

**Proof** We may express

$$V(a, a+b) = \sum_{u=0}^b (-1)^u \binom{b}{u} V(a+u, a+u) \quad (55)$$

from the binomial theorem, given the definition  $V(a, b) = (q-1)^a q^{(d+1)s-b}$ .

From the definition we have

$$\Lambda_\Gamma(\mathbf{a}, \sigma) = \mathcal{G}_\Gamma(\mathbf{a}, \sigma) - \sum_{\sigma^{(1)}} \mathcal{G}_\Gamma(\mathbf{a}, \sigma^{(1)}) + \dots + (-1)^i \sum_{\sigma^{(i)}} \mathcal{G}_\Gamma(\mathbf{a}, \sigma^{(i)}) + \dots,$$

where the sums run over  $\sigma^{(i)} \subseteq \sigma$  obtained from  $\sigma$  by replacing exactly  $i$  occurrences of  $\square$  by  $*$ . We will apply Proposition 17 to evaluate these terms, then simplify.

If the sequence  $\circ\square$  appears within  $\sigma$ , then we call signature  $\Gamma$ -*non-strict*, as any corresponding short pattern  $\mathfrak{T}$  with  $\sigma \subseteq \sigma_{\mathfrak{T}}$  is non-strict, and by definition  $\mathcal{G}_{\Gamma}(\mathfrak{T}, \sigma) = 0$ . Thus the alternating sum for  $\Lambda_{\Gamma}$  will only contain non-zero contributions from subsignatures when all such  $\square$ 's occurring as part of a  $\circ\square$  string in  $\sigma$  have been removed (i.e. changed to an  $*$ ). Upon doing this, the signature will no longer possess any subwords of the form  $\circ\square$ , and we may again apply the above formula for  $G_{\Gamma}$  to these subsignatures. This is reflected in the definition of  $a_{\Gamma}$  and in the statement of the Theorem.

We first assume that  $\sigma_0 \neq \square$  and that  $\circ\square$  does not occur within  $\sigma$ . Then

$$\begin{aligned} G_{\Gamma}(\mathbf{a}, \sigma) &= \chi_{\Gamma} V(a_{\Gamma}, a_{\Gamma} + d_{\Gamma}) = \chi_{\Gamma} \sum_{u=0}^{d_{\Gamma}} (-1)^u \binom{d_{\Gamma}}{u} V(a_{\Gamma} + u) \\ &= \chi_{\Gamma} \sum_{u=0}^{d_{\Gamma}} (-1)^u V(a_{\Gamma} + u) \sum_{l=0}^{k_{\Gamma}} \sum_{\{i_1, \dots, i_l\}: \sigma_{i_j} = \square} \binom{k_{\Gamma}}{u-l} \delta_n(i_1, \dots, i_l) \\ &= \chi_{\Gamma} \sum_{l=0}^{k_{\Gamma}} \sum_{\{i_1, \dots, i_l\}: \sigma_{i_j} = \square} \delta_n(i_1, \dots, i_l) \sum_{u=l}^{d_{\Gamma}} (-1)^u \binom{k_{\Gamma}}{u-l} V(a_{\Gamma} + u) \end{aligned}$$

where we have used Lemmas 17 and 9, resp., in the first two steps, and in the last step have simply interchanged the order of summation.

By similar calculation (still assuming that  $\sigma_0 \neq \square$  for simplicity of exposition) we have

$$\begin{aligned} \sum_{\sigma^{(m)} \subseteq \sigma} G_{\Gamma}(\mathbf{a}, \sigma^{(m)}) &= \chi_{\Gamma} \sum_{\{i_1, \dots, i_m\}: \sigma_{i_j} = \square} \delta_n(i_1, \dots, i_m) \sum_{l=0}^{k_{\Gamma}-m} \sum_{\{i'_1, \dots, i'_l\}: \sigma_{i'_j} \neq \sigma_{i_j}} \delta_n(i_1, \dots, i_l) \\ &\quad \times \sum_{u=l}^{d_{\Gamma}-2m} (-1)^u \binom{k_{\Gamma}-m}{u-l} V(a_{\Gamma} + 2m + u) \end{aligned}$$

where we can write the upper bound on the sum over  $u$  as an absolute constant, since either the divisibility conditions are satisfied and the upper bound (equal to  $d_{\Gamma}(\sigma^{(m)})$ ) is indeed  $d_{\Gamma}(\sigma) - 2m$  or else the term is 0. Simplifying by combining the

two sums with divisibility conditions, we have

$$\begin{aligned} \sum_{\sigma^{(m)} \subseteq \sigma} G_{\Gamma}(\mathbf{a}, \sigma^{(m)}) &= \chi_{\Gamma} \sum_{l=0}^{k_{\Gamma}-m} \sum_{\{i_1, \dots, i_{m+l}\}: \sigma_{i_j} = \square} \binom{m+l}{m} \delta_n(i_1, \dots, i_{m+l}) \\ &\quad (-1)^l \sum_{v=0}^{d_{\Gamma}-2m-l} (-1)^v \binom{k_{\Gamma}-m}{v} V(a_{\Gamma} + 2m + l + v) \end{aligned}$$

Hence

$$\begin{aligned} \Lambda_{\Gamma}(\mathbf{a}, \sigma) &= \chi_{\Gamma} \sum_{m=0}^{k_{\Gamma}} (-1)^m \sum_{l=0}^{k_{\Gamma}-m} \sum_{\{i_1, \dots, i_{m+l}\}: \sigma_{i_j} = \square} \binom{m+l}{m} \delta_n(i_1, \dots, i_{m+l}) \\ &\quad (-1)^l \sum_{v=0}^{d_{\Gamma}-2m-l} (-1)^v \binom{k_{\Gamma}-m}{v} V(a_{\Gamma} + 2m + l + v) \\ &= \chi_{\Gamma} \sum_{m=0}^{k_{\Gamma}} \sum_{x=m}^{k_{\Gamma}} \sum_{\{i_1, \dots, i_x\}: \sigma_{i_j} = \square} \binom{x}{m} \delta_n(i_1, \dots, i_x) (-1)^x \\ &\quad \sum_{v=0}^{d_{\Gamma}-x-m} (-1)^v \binom{k_{\Gamma}-m}{v} V(a_{\Gamma} + m + x + v) \\ &= \chi_{\Gamma} \sum_{x=0}^{k_{\Gamma}} \sum_{\{i_1, \dots, i_x\}: \sigma_{i_j} = \square} \delta_n(i_1, \dots, i_x) (-1)^x \\ &\quad \sum_{m=0}^x \binom{x}{m} \sum_{v=0}^{d_{\Gamma}-x-m} (-1)^v \binom{k_{\Gamma}-m}{v} V(a_{\Gamma} + m + x + v) \end{aligned} \quad (56)$$

where in the first step we changed the sum over  $l$  to a sum over  $x = m + l$  and interchanged the order of summation in the second step. Now let  $w = m + v$ , so (56) equals

$$\begin{aligned} \sum_{m=0}^x (-1)^m \sum_{w=m}^{d_{\Gamma}-x} (-1)^w \binom{x}{m} \binom{k_{\Gamma}-m}{w-m} V(a_{\Gamma} + x + w) &= \\ \sum_{w=0}^{d_{\Gamma}-x} (-1)^w V(a_{\Gamma} + x + w) \sum_{m=0}^w (-1)^m \binom{x}{m} \binom{k_{\Gamma}-m}{w-m} \end{aligned} \quad (57)$$

But

$$\sum_{m=0}^w (-1)^m \binom{x}{m} \binom{k_{\Gamma}-m}{w-m} = \binom{k_{\Gamma}-x}{w} \quad (58)$$

so combining (56) and (57) and applying (58)

$$\Lambda_\Gamma(\mathbf{a}, \sigma) = \chi_\Gamma \sum_{m=0}^{k_\Gamma} (-1)^m \sum_{\{i_1, \dots, i_{m+1}\}: \sigma_{i_j} = \square} \delta_n(i_1, \dots, i_x) (-1)^x \sum_{w=0}^{d_\Gamma - x} (-1)^w \binom{k_\Gamma - x}{w} V(a_\Gamma + x + w)$$

The cases where  $\sigma_0 = \square$  or where  $\circ \square$  appears in the signature follow by a straightforward generalization.  $\square$

We turn now to the evaluation of  $\Lambda_\Delta(\mathbf{a}', \sigma)$ , where  $\sigma$  is unchanged and

$$\mathbf{a}' = \left\{ \begin{array}{cccccc} \beta_1 & \beta_2 & \cdots & \beta_d & s \\ & \alpha_1 & & \alpha_d & \end{array} \right\}.$$

Let

$$\begin{aligned} \varepsilon_\Delta(\sigma) &= \varepsilon_\Delta = \begin{cases} 1 & \text{if } \sigma_d = \square, \\ 0 & \text{otherwise,} \end{cases} \\ \mathcal{K}_\Delta &= \{0 < i \leq d \mid \sigma_{i-1} = \square, \sigma_i \neq \circ\}, \quad k_\Delta(\sigma) = k_\Delta = |\mathcal{K}_\Delta|, \\ \mathcal{N}_\Delta &= \{0 < i \leq d \mid \sigma_{i-1} = \square, \sigma_i = \circ\}, \quad n_\Delta(\sigma) = n_\Delta = |\mathcal{N}_\Delta|, \\ \mathcal{C}_\Delta &= \{1 \leq i \leq d \mid \sigma_i, \sigma_{i+1}, \dots, \sigma_d \text{ not all } \circ \text{ and either } \sigma_{i-1} = * \text{ or } i \in \mathcal{N}_\Delta\}, \\ \chi_\Delta(\mathbf{a}, \sigma) &= \chi_\Delta = \prod_{i \in \mathcal{C}_\Delta(\sigma)} \delta_n(i), \quad t_\Delta = |\{0 \leq i < d \mid \sigma_i = \eta_i\}| \end{aligned}$$

$$a_\Delta(\sigma) = a_\Delta = 2(d - (t_\Delta - n_\Delta)) + \begin{cases} -1 & \text{if } \sigma_d = \circ \\ 0 & \text{if } \sigma_d = \square \\ 1 & \text{if } \sigma_d = * \end{cases} + \begin{cases} 1 & \text{if } \sigma_0 = \circ \\ 0 & \text{if } \sigma_0 \neq \circ \end{cases}.$$

We give  $\delta_n(i_1, \dots, i_n)$  the same meaning as before: it is  $\delta_n(i_1, \dots, i_k; \mathbf{a})$ . But since the top row of  $\mathbf{a}'$  is in terms of the  $\beta$ 's, it is worth noting that it can also be described as 1 if  $n$  divides  $\beta_{i_1}, \dots, \beta_{i_k}$  and 0 otherwise. Indeed,  $n|s$  so  $n|\alpha_i$  if and only if  $n|\beta_i = s - \alpha_i$ .

**Theorem 7** *With notation as above we have*

$$\Lambda_\Delta(\mathbf{a}', \sigma) = (-1)^{n_\Delta + \varepsilon_\Delta} \chi_\Delta \sum_{x=0}^{k_\Delta} \sum_{\substack{\Sigma \subseteq \mathcal{K}_\Delta \\ |\Sigma| = x}} \delta_n(\Sigma) (-1)^x V(a_\Delta + x, a_\Delta + k_\Delta).$$

where (as defined above) the inner sums range over subsets of  $\mathcal{K}_\Delta$ .

**Proof** We can reuse our previous work by noting that

$$\Lambda_\Delta(\mathbf{a}', \sigma) = \Lambda_\Gamma(\tilde{\mathbf{a}}, \tilde{\sigma}),$$

where

$$\tilde{\mathbf{a}} = \left\{ \begin{array}{ccccccc} s & \beta_d & \beta_{d-1} & \cdots & \beta_1 \\ & \alpha_d & \alpha_{d-1} & \cdots & \alpha_1 \end{array} \right\}, \quad \tilde{\sigma} = \sigma_d \sigma_{d-1} \cdots \sigma_0.$$

Roughly speaking we can just take the mirror image of our previous formula. But there is one point of caution: in going from  $\mathbf{a}$  to  $\tilde{\mathbf{a}}$  we reflected  $\sigma$  in the range 0 to  $d$ , while we reflected  $\alpha$  in the range 1 to  $d$  (and changed it to  $\beta$ , which has no effect on  $\delta$ ). This means the  $\mathcal{C}_\Delta(\sigma)$ , if it is to be the set of locations where the congruences are taken in evaluating  $\delta$ , is *not* the mirror image of  $\mathcal{C}_\Gamma(\tilde{\mathbf{a}})$  in the range 0 to  $d$ , but the shift of that mirror image to the right by 1, which makes  $\mathcal{C}_\Delta(\sigma)$ , like  $\mathcal{C}_\Gamma(\sigma)$ , a subset of the range from 1 to  $d$ . There are corresponding adjustments in the definitions of  $\mathcal{K}_\Delta$  and  $\mathcal{N}_\Delta$ .  $\square$

Let  $\mathcal{A}_\Gamma(\sigma)$  denote the set of  $\Gamma$ -admissible sets for  $\sigma$ , and we let  $\mathcal{A}_\Delta(\sigma)$  denote the set of  $\Delta$ -admissible sets. Let  $\Pi$  be an  $f$ -packet (as defined in Section 3 before Statement F). By Theorems 6 and 7 we may reformulate Statement F in the following way.

**Statement G.** *With notation as in Section 11, we have*

$$\begin{aligned} \sum_{\sigma} (-1)^{n_\Gamma(\sigma) + \varepsilon_\Gamma(\sigma)} \sum_{\substack{0 \leq x \leq k_\Gamma(\sigma) \\ \Sigma \in \mathcal{A}_\Gamma(\sigma) \\ |\Sigma - \mathcal{C}_\Gamma(\sigma)| = x}} (-1)^x V(a_\Gamma + x, a_\Gamma + k_\Gamma) \delta_n(\Sigma; \mathbf{a}) = \\ \sum_{\sigma} (-1)^{n_\Delta(\sigma) + \varepsilon_\Delta(\sigma)} \sum_{\substack{0 \leq x \leq k_\Gamma(\sigma) \\ \Sigma \in \mathcal{A}_\Gamma(\sigma) \\ |\Sigma - \mathcal{C}_\Gamma(\sigma)| = x}} (-1)^x V(a_\Delta + x, a_\Delta + k_\Delta) \delta_n(\Sigma; \mathbf{a}) \end{aligned} \quad (59)$$

The outer sum is over  $f$ -subsignatures  $\sigma$  of  $\eta$ , since in (27) each such signature appears exactly once on each side. We recall that the packet  $\Pi$  in Statement F intersects each open  $f$ -facet  $\mathcal{S}_\sigma$  in a unique element  $\mathbf{a}$ , and so  $\mathbf{a}$  is determined by  $\sigma$ . We have restored  $\mathbf{a}$  to the notation  $\delta_n(\Sigma; \mathbf{a})$  from which it was suppressed in Theorems 6 and 7, because the dependence of these terms on  $\mathbf{a}$  – or, equivalently, on  $\sigma$  – will now become our most important issue.

## 12 Concurrency

This section contains purely combinatorial results that are needed for the proof. The motivation of these results comes from the appearance of divisibility conditions through the factor  $\delta_n(\Sigma; \mathbf{a})$  defined in (52) that appears in Theorems 6 and 7. We refer to the discussion of Statement G in Section 3 for the context of the results of this Section.

Let  $0 \leq f \leq d$ . In Section 3 we defined bijections  $\phi_{\sigma, \tau} : \mathcal{S}_\sigma \longrightarrow \mathcal{S}_\tau$  between the open  $f$ -facets, and a related equivalence relation, whose classes we call  $f$ -packets. According to Statement F, the sum of  $\Lambda_\Gamma(\mathbf{a}, \sigma)$  over an  $f$ -packet is equal to the corresponding sum of  $\Lambda_\Delta(\mathbf{a}', \sigma)$ . Moreover in Theorems 6 and 7, we have rewritten  $\Lambda_\Gamma$  and  $\Lambda_\Delta$  as sums over ordered subsets of  $\mathcal{K}_\Gamma$  and  $\mathcal{K}_\Delta$ . In order to prove Statement F, we will proceed by identifying terms in the resulting double sum that can be matched, and that is the aim of the results of this section.

**Definition 1 (Concurrency)** *Let  $\sigma$  and  $\tau$  be subsignatures of  $\eta$  that have the same number of  $*$ 's. Fix two subsets  $\Sigma = \{j_1, \dots, j_l\}$  and  $\Sigma' = \{j'_1, \dots, j'_l\}$  of  $\{1, 2, \dots, d\}$  of equal cardinality, and arranged in ascending order:*

$$0 \leq j_1 < j_2 < \dots < j_l \leq d, \quad 0 \leq j'_1 < j'_2 < \dots < j'_l \leq d.$$

*We say that the pairs  $(\sigma, \Sigma)$  and  $(\tau, \Sigma')$  concur if the following conditions are satisfied. We require that for  $1 \leq m \leq l$  the two sets*

$$\{t \mid j_m \leq t \leq d, \sigma_t = *\}, \quad \{t \mid j'_m \leq t \leq d, \tau_t = *\} \quad (60)$$

*have the same cardinality, and that  $\eta_i = \circ$  for*

$$\min(j_m, j'_m) \leq i < \max(j_m, j'_m). \quad (61)$$

Concurrency is an equivalence relation.

**Example 1** Let  $\eta = (\eta_0, \eta_1, \dots, \eta_5) = (\circ, \square, \circ, \circ, \square, \square)$ . The pairs

$$((*, \square, *, \circ, *, \square), \{2, 4, 5\}); \quad ((\circ, *, *, *, \square, \square), \{2, 3, 5\})$$

concur. However

$$((*, \square, *, \circ, *, \square), \{2, 4, 5\}); \quad ((\circ, *, *, *, \square, \square), \{2, 4, 5\})$$

do not, as the number of  $*$ 's to the right of  $\sigma_4, \tau_4$  differ.

**Proposition 18** *Suppose that the pairs  $(\sigma, \Sigma)$  and  $(\tau, \Sigma')$  concur. Then if  $\phi_{\sigma, \tau}(\mathbf{a}) = \mathbf{b}$ , where*

$$\mathbf{a} = \left\{ \begin{array}{cccccc} s & \alpha_1 & \cdots & \alpha_d \\ \beta_1 & & \cdots & \beta_d \end{array} \right\}, \quad \mathbf{b} = \left\{ \begin{array}{cccccc} s & \mu_1 & \cdots & \mu_d \\ \nu_1 & & \cdots & \nu_d \end{array} \right\},$$

we have  $\alpha_{j_m} = \mu_{j'_m}$  ( $1 \leq m \leq l$ ).

This implies that

$$\delta_n(\Sigma; \mathbf{a}) = \delta_n(\Sigma'; \mathbf{b}),$$

which can be used to compare the contributions of these ordered subsets to  $\Lambda_\Gamma(\mathbf{a}, \sigma)$  and  $\Lambda_\Delta(\mathbf{a}', \sigma)$  with the corresponding contributions to  $\Lambda_\Gamma(\mathbf{b}, \tau)$  and  $\Lambda_\Delta(\mathbf{b}', \tau)$  in the formulas of Theorems 6 and 7.

**Proof** It is sufficient to check this when  $\mathbf{a}$  is a vertex of  $\overline{\mathcal{S}_\sigma}$ . Indeed, both  $\alpha_{j_m}$  and  $\mu_{j'_m}$  are affine-linear functions of  $\alpha_1, \dots, \alpha_d$ , so if they are the same when  $\mathbf{a} = \mathbf{a}_k$  is a vertex, they will be the same for convex combinations of the vertices, that is, for all elements of  $\overline{\mathcal{S}_\sigma}$ . Because  $\mathbf{a}_k$  is a vertex of  $\overline{\mathcal{S}_\sigma}$ ,  $\sigma_k = *$ ; if  $\sigma_k$  is the  $r$ -th  $*$  in  $\sigma$ , then by definition  $\phi_{\sigma, \tau}(\mathbf{a}) = \mathbf{a}_l$  where  $\tau_l$  is the  $r$ -th  $*$  in  $\tau$ . This is a consequence of the definition of  $\phi_{\sigma, \tau}$ . Now our assumption on the cardinality of the two sets (60) implies that  $k \leq j_m$  if and only if  $l \leq j'_m$ .

Now we prove that  $\alpha_{j_m} = \mu_{j'_m}$ . There are now two cases, depending on whether  $j_m \leq k$  (and so  $j'_m \leq l$ ) or not. First suppose that  $j_m \leq k$  and  $j'_m \leq l$ . Then we have  $\alpha_i - \alpha_{i+1} = c'_i$  for all  $i$  except  $k$  and  $\mu_i = \mu_{i+1} = c'_i$  for all  $i$  except  $l$ , and  $\alpha_0 = s = \mu_0$ . This means that  $\alpha_i = \mu_i$  when  $i \leq \min(k, l)$ , *a fortiori* when  $i \leq \min(j_m, j'_m)$ . Suppose for definiteness that  $j_m \leq j'_m$ , so  $\min(j_m, j'_m) = j_m$ . Thus we have proved that  $\alpha_{j_m} = \mu_{j_m}$ . Since by hypothesis  $\eta_{j_m} = \eta_{j_m+1} = \dots = \eta_{j'_m-1} = \circ$  we also have  $\mu_{j_m} = \mu_{j_m+1} = \dots = \mu_{j'_m}$  and herefore  $\alpha_{j_m} = \mu_{j'_m}$ . The case where  $j_m \geq j'_m$  is similar, and the case where  $j_m \leq k$  and  $j'_m \leq l$  is settled.

Next suppose that  $j_m > k$  and so  $j'_m > l$ . Then  $\alpha_i - \alpha_{i+1} = c'_i$  for all  $i$  except  $k$  and  $\mu_i = \mu_{i+1} = c'_i$  for all  $i$  except  $l$ , and  $\alpha_{d+1} = 0 = \mu_{d+1}$ , we get  $\alpha_i = \mu_i$  for  $i > \max(k, l)$ , *a fortiori* for  $i > \max(j_m, j'_m)$ . Suppose for definiteness that  $j_m \leq j'_m$ , so that  $\max(j_m, j'_m) = j'_m$ . We have proved that  $\alpha_{j'_m} = \mu_{j'_m}$ . Our hypothesis that  $\eta_{j_m} = \eta_{j_m+1} = \dots = \eta_{j'_m-1} = \circ$  implies that  $\alpha_{j_m} = \alpha_{j_m+1} = \dots = \alpha_{j'_m}$ , and so we get  $\alpha_{j_m} = \mu_{j'_m}$ . The case  $j_m \geq j'_m$  is again similar.  $\square$

We now introduce certain operations on signatures that give rise to concurrences.

**Definition 2 ( $\Gamma$ - and  $\Delta$ -swaps)** *Let  $\sigma$  and  $\tau$  be subsignatures of  $\eta$ . We say that  $\tau$  is obtained from  $\sigma$  by a  $\Gamma$ -swap at  $i - 1, i$  if*

$$\sigma_j = \tau_j \quad \text{for all } j \neq i - 1, i, \quad \sigma_{i-1} = *, \sigma_i = \square, \quad \tau_{i-1} = \circ, \tau_i = *,$$

and by a  $\Delta$ -swap at  $i - 1, i$  if

$$\sigma_j = \tau_j \quad \text{for all } j \neq i - 1, i, \quad \sigma_{i-1} = \square, \sigma_i = *, \quad \tau_{i-1} = *, \tau_i = \circ.$$

**Definition 3 ( $\Gamma$ - and  $\Delta$ -admissibility)** We say that a subset  $\Sigma = \{j_1, j_2, \dots, j_m\}$  of  $\{1, 2, 3, \dots, d\}$  is  $\Gamma$ -admissible for  $\sigma$  if

$$\mathcal{C}_\Gamma(\sigma) \subset \Sigma \subset \mathcal{C}_\Gamma(\sigma) \cup \mathcal{K}_\Gamma(\sigma),$$

and similarly it is  $\Delta$ -admissible if  $\mathcal{C}_\Delta(\sigma) \subset \Sigma \subset \mathcal{C}_\Delta(\sigma) \cup \mathcal{K}_\Delta(\sigma)$ .

**Proposition 19 (Swapped data concur)** (a) Suppose  $\tau$  is obtained from a  $\Gamma$ -swap at  $i - 1, i$ . Assume that  $i \notin \Sigma$ . Let  $0 < j_1 < j_2 < \dots < j_l \leq d$  be a sequence such that  $j_m \neq i$  for all  $m$ . Let

$$j'_m = \begin{cases} j_m & \text{if } j_m \neq i - 1; \\ i & \text{if } j_m = i - 1. \end{cases}$$

Then  $(\sigma, \Sigma)$  and  $(\tau, \Sigma')$  concur, where  $\Sigma = \{j_1, \dots, j_m\}$  and  $\Sigma' = \{j'_1, \dots, j'_m\}$ . Moreover  $\Sigma$  is  $\Gamma$ -admissible for  $\sigma$  if and only if  $\Sigma'$  is  $\Gamma$ -admissible for  $\tau$ .

(b) Suppose that  $\tau$  is obtained from a  $\Delta$ -swap at  $i - 1, i$ . Assume that  $i \notin \Sigma$ . Let  $0 < j_1 < j_2 < \dots < j_l \leq d$  be a sequence such that  $j_m \neq i$  for all  $m$ . Let

$$j'_m = \begin{cases} j_m & \text{if } j_m \neq i + 1; \\ i & \text{if } j_m = i + 1. \end{cases}$$

Then  $(\sigma, \Sigma)$  and  $(\tau, \Sigma')$  concur, where  $\Sigma = \{j_1, \dots, j_m\}$  and  $\Sigma' = \{j'_1, \dots, j'_m\}$ . Moreover  $\Sigma$  is  $\Delta$ -admissible for  $\sigma$  if and only if  $\Sigma'$  is  $\Delta$ -admissible for  $\tau$ .

**Proof** This is straightforward to check from the definitions of concurrence and admissibility. One point merits further discussion. Suppose we are in case (a) for definiteness. If  $\Sigma$  is  $\Gamma$ -admissible, then according to the definition (51),  $i - 1 \in \mathcal{C}_\Gamma(\sigma) \subseteq \Sigma$  if  $\sigma_0, \dots, \sigma_{i-2}$  are not all  $\circ$ . In this case,  $i \in \mathcal{C}_\Gamma(\tau) \subseteq \Sigma'$ . If instead,  $\sigma_0 = \dots = \sigma_{i-2} = \circ$ , then  $i - 1 \notin \Sigma$  but then under the  $\Gamma$ -swap,  $\tau_0 = \dots = \tau_{i-1} = \circ$  and so  $i \notin \Sigma'$ .  $\square$

If the hypotheses of Proposition 19 are satisfied we say that  $(\tau, \Sigma')$  is obtained from  $(\sigma, \Sigma)$  by a  $\Gamma$ -swap (or  $\Delta$ -swap).

Let us define an equivalence relation on the set of pairs  $(\sigma, \Sigma)$ , where  $\sigma$  is a subsignature of  $\eta$  and  $\Sigma$  is a  $\Gamma$ -admissible subset of  $\{1, 2, \dots, d\}$ .

**Definition 4 ( $\Gamma$ - and  $\Delta$ -packs)** We write  $(\sigma, \Sigma) \sim_{\Gamma} (\tau, \Sigma)$  if  $(\tau, \Sigma)$  can be obtained by a sequence of  $\Gamma$ -swaps or inverse  $\Gamma$ -swaps. We call an equivalence class a  $\Gamma$ -pack; and  $\Delta$ -packs are defined similarly.

**Lemma 10** Each  $\Gamma$ -pack or  $\Delta$ -pack contains a unique element with maximal number of  $\circ$ 's. Within the pack, this unique element  $(\sigma, \Sigma)$  is characterized as follows.

$\Gamma$ -pack:           Whenever  $\eta_{i-1}\eta_i = \circ\square$  and  $\sigma_{i-1}\sigma_i = *\square$  we have  $i \in \Sigma$ ,  
 $\Delta$ -pack:           Whenever  $\eta_{i-1}\eta_i = \square\circ$  and  $\sigma_{i-1}\sigma_i = \square*$  we have  $i \in \Sigma$ .

**Proof** If  $\eta_{i-1}\eta_i = \circ\square$  and  $\sigma_{i-1}\sigma_i = *\square$  then a  $\Gamma$ -swap is possible at  $i-1, i$  if and only if  $i \notin \Sigma$ . Indeed, the fact that  $\Sigma$  is  $\Gamma$ -admissible for  $\sigma$  means that  $i-1 \in \Sigma$ . This assertion therefore follows from Proposition 19.

Clearly the element maximizing the number of  $\circ$ 's is obtained by making all possible swaps. The statements are now clear for the  $\Gamma$ -pack, and for the  $\Delta$  pack they are similar.  $\square$

**Definition 5 (Origins)** We call the unique element with the greatest number of  $\circ$ 's the origin of the pack. We say that  $(\sigma, \Sigma)$  is a  $\Gamma$ -origin if it is the origin of its  $\Gamma$ -pack, and  $\Delta$ -origins are defined the same way.

As we have explained, our goal is to exhibit a bijection  $\psi$  between the  $\Gamma$ -packs and the  $\Delta$ -packs. It will be sufficient to exhibit a bijection between their origins. Let  $(\sigma, \Sigma)$  be the origin of a  $\Gamma$ -pack; we will denote  $\psi(\sigma, \Sigma) = (\sigma', \Sigma')$ . We can define  $\sigma'$  immediately. To obtain  $\sigma'$ , we break  $\eta$  (which involves only  $\square$ 's and  $\circ$ 's) into maximal strings of the form  $\circ \cdots \circ$  and  $\square \cdots \square$ , and we prescribe  $\sigma'$  on these ranges.

- ( **$\circ$ 's in  $\sigma$  reflect across the midpoint of the string of  $\circ$ 's in  $\eta$** ) Let  $\eta_h, \dots, \eta_k$  be a maximal consecutive string of  $\circ$ 's in  $\eta$  (so  $\eta_{h-1}, \eta_{k+1} \neq \circ$ ). If  $h \leq i \leq k$  then  $\sigma'_i = \sigma_{h+k-i}$ .
- (**Distinguished  $\square$ 's in  $\sigma$  slide one index leftward**) Let  $\sigma_h \dots \sigma_k$  be a maximal consecutive string of  $\square$ 's in  $\sigma$  (so  $\sigma_{h-1}, \sigma_{k+1} \neq \square$ ). Let  $h \leq i \leq k$  be the smallest element of  $\Sigma$  in this range, or if none exists, let  $i = k+1$ . Then if  $\eta_{h-1} = \square$  and  $\sigma_{h-1} = *$  then  $\psi(\sigma) = \sigma'$  has  $\sigma'_{h-1} = \cdots \sigma'_{i-2} = \square$ ,  $\sigma'_{i-1} = *$ , and  $\sigma'_i = \cdots = \sigma'_k = \square$ . If either  $\eta_{h-1} = \circ$  or  $\sigma_{h-1} \neq *$ , then  $\psi$  leaves the string of  $\square$ 's in  $\sigma$  unchanged.

The last rule merits further explanation. Since  $\sigma$  is a subsignature of  $\eta$ , the maximal chain  $\sigma_h \cdots \sigma_k$  of boxes in  $\sigma$  is contained in a (usually longer) maximal chain of boxes  $\eta_l \eta_{l+1} \cdots \eta_m$  within  $\eta$ ; thus  $l \leq h$  and  $m \geq k$  and the range from  $l$  to  $m$  is thus broken up into smaller ranges of which  $\sigma_h \cdots \sigma_k = \square \cdots \square$  is one. We assume that  $\sigma_{h-1} = *$  and that  $\eta_{h-1} = \square$ . In this case we will modify  $\sigma_h \cdots \sigma_k$ . But if the condition that  $\sigma_{h-1} = *$  and that  $\eta_{h-1} = \square$  is not met, we leave it unchanged – and the condition will be met if and only if  $h > l$ . Then with  $i$  as in the second rule above, we make the following shift:

$$\left\{ \begin{array}{cccccc} \sigma_{h-1} & \sigma_h & \cdots & \sigma_{i-1} & \cdots & \sigma_k \\ * & \square & \cdots & \square & \cdots & \square \end{array} \right\} \longrightarrow \left\{ \begin{array}{cccccc} \sigma'_{h-1} & \sigma'_h & \cdots & \sigma'_{i-1} & \cdots & \sigma'_k \\ \square & \square & \cdots & * & \cdots & \square \end{array} \right\}. \quad (62)$$

It is useful to divide up the nodal signature  $\eta$  into blocks of consecutive  $\square$ 's alternating with blocks of consecutive  $\circ$ 's (where a block might consist of just one of these characters), e.g.

$$\eta = (\eta_0, \eta_1, \dots, \eta_7) = (\square, \circ, \circ, \underbrace{\square, \square, \square}_{\text{block of } \square\text{'s}}, \circ, \circ, ).$$

Formally, a  $\square$ -block is a maximal consecutive set  $B = \{h, h+1, \dots, k\}$  such that  $\eta_i$  are all  $\square$ 's, and  $\circ$ -blocks are defined similarly. The map  $\psi$  can be understood according to what it does to the indices of  $\sigma$  contained within each of these blocks (and no two indices from different blocks interact under  $\psi$ ). In particular, the number of  $*$ 's in  $\sigma$  contained within a block of  $\eta$  is preserved under  $\psi$ . We use this fact repeatedly in the proofs, as it often implies that it is enough to work locally within a block of  $\square$ 's or  $\circ$ 's.

We have not yet described what  $\psi$  does to  $\Sigma$ . The next result will make this possible. Define

$$\begin{aligned} P_\sigma(u) &= |\{j \geq u \mid \sigma_j = *\}|, \\ Q_\sigma(u) &= |\{j \geq u \mid \sigma_j = \square\}|. \end{aligned}$$

If  $u, v \in \{1, 2, \dots, d\}$  then we say that the pair  $(u, v)$  is *equalized* for  $\sigma$  and  $\sigma'$  if

$$P_\sigma(u) = P_{\sigma'}(v), \quad Q_\sigma(u) = Q_{\sigma'}(v). \quad (63)$$

**Lemma 11** *Let  $\sigma, \sigma'$  be signatures with  $\psi(\sigma') = \sigma$ .*

(i) *If  $1 \leq u \leq d$  and  $\eta_u \neq \eta_{u-1}$  then  $(u, u)$  is equalized.*

(ii) *Suppose that  $B$  is a  $\circ$ -block and that  $u \in B$  such that  $\sigma_u = *$ . Assume that  $\sigma_j \neq \circ$  for some  $j < u$ . Then there exists  $0 < v \in B$  such that  $\sigma'_v = *$  and  $(u, v)$  is equalized.*

(iii) Suppose that  $B$  is a  $\circ$ -block and that  $v \in B$  such that  $\sigma'_v = *$  but  $\sigma'_{v-1} \neq \circ$ . Then there exists  $0 < u \in B$  such that  $(u, v)$  is equalized,  $\sigma_u = *$  and  $\sigma_j \neq \circ$  for some  $j < u$ .

(iv) Given  $i$  as in the second rule for  $\psi$  on signatures, the pair  $(i, i)$  is equalized.

The condition in (ii) and (iii) that  $\sigma_u = *$  and  $\sigma_j \neq \circ$  for some  $j < u$  means that  $u \in \mathcal{C}_\Gamma(\sigma)$ .

**Proof** Part (i) follows from the fact that  $u$  is at the left edge of a block when  $\eta_u \neq \eta_{u-1}$ . Indeed, if  $B$  is a  $\square$ - or  $\circ$ -block then  $\sigma$  and  $\sigma'$  have the same number of  $*$ 's and  $\square$ 's in  $B$ . Since  $u$  is at the left edge of a block, then the accumulated numbers of  $*$  and  $\square$  in that block and those to the right are the same for  $\sigma$  and  $\sigma'$  and so  $(u, u)$  is equalized.

To prove (ii), observe that the number of  $*$  in the  $\circ$ -block  $B = \{h, h+1, \dots, k\}$  are the same, and  $(k+1, k+1)$  are equalized (or else  $k = d$ ), so counting from the right, if  $\sigma_u$  is the  $r$ -th  $*$  within the block, we can take  $\sigma'_v$  to be the  $r$ -th  $*$  for  $\sigma'$  in the block, and we have equalization. The hypothesis that  $\sigma_u \neq \circ$  for some  $j < u$  guarantees that either  $B$  is not the first block, or that  $\sigma_u$  is not the leftmost  $*$  in the block, so  $v > 0$ .

To prove (iii), we argue the same way, and the only thing to be checked is that  $j > 0$  and that  $\sigma_j \neq \circ$  for some  $j < u$ . This follows from the assumption that  $\sigma'_{v-1} \neq \circ$ , since if  $\sigma'_{v-1} = \square$  then  $B$  is not the first block, while if  $\sigma'_{v-1} = *$  then  $\sigma'_v$  is not the first  $*$  in  $B$  for  $\sigma'$ , hence also not the first  $*$  in  $B$  for  $\sigma$ .

For (iv), the  $\square$ -block containing  $i$  can be broken up into segments of the form  $*\square \cdots \square$  as in the left-hand side of (62) and possibly an initial string consisting entirely of  $\square$ 's. According to the second rule for  $\psi$  on signatures, the image of each such segment under  $\psi$  also contains exactly one  $*$  (excluding the possible initial string of  $\square$ 's without  $*$ 's) and the same number of  $\square$ 's. As  $i$  occurs to the right of both the  $*$  in  $\sigma$  and  $\sigma'$  in the respective segments as depicted in (62), it is thus clear that  $(i, i)$  is equalized.  $\square$

**Proposition 20 (Concurrence of origins)** *Let  $(\sigma, \Sigma)$  be the origin of a  $\Gamma$ -pack, and let  $\sigma' = \psi(\sigma)$ . Given any  $j \in \Sigma$  we can associate a corresponding index  $j' \in \Sigma'$  as follows. There exists a unique  $1 \leq t = t(j) \leq d$  such that  $(j, t)$  are equalized, and such that  $\sigma'_t \neq \circ$ . Define  $j' = \psi(j)$  so that  $j' - 1$  is the largest index  $< t(j)$  such that  $\sigma'_{j'-1} \neq \circ$ . Then the  $\Sigma' = \{\psi(j) | j \in \Sigma\}$  is  $\Delta$ -admissible for  $\sigma'$ , and in fact  $(\sigma', \Sigma')$  is a  $\Delta$ -origin. Moreover the pairs  $(\sigma, \Sigma')$  and  $(\sigma', \Sigma')$  concur. The map  $\psi : (\sigma, \Sigma) \mapsto (\sigma', \Sigma')$  is a bijection from the set of  $\Gamma$ -origins to the set of  $\Delta$ -origins.*

Before proving this, we give several examples.

1. If  $\eta = (\circ, \circ, \circ, \circ, \circ, \circ, \square)$ ,

$$\psi((\circ, \circ, *, \circ, *, *, *), \{4, 5, 6\}) \mapsto ((*, *, \circ, *, \circ, \circ, *), \{1, 2, 4\})$$

Indeed,  $\psi$  reflects all entries in the initial block of 6  $\circ$ 's in  $\eta$ . In the block consisting of a single  $\square$  at the end of  $\eta$ ,  $\sigma$  contains no  $\square$ 's and so  $\sigma'$  agrees with  $\sigma$  on this block. The reader will check that that  $t(6) = 4, t(5) = 2$ , and  $t(4) = 1$ . Thus  $\Sigma'$  is as defined in the Proposition. To check that  $\Sigma$  is  $\Gamma$ -admissible for  $\sigma$ , note that  $\mathcal{C}_\Gamma(\sigma) = \{4, 5, 6\}$  and  $\mathcal{K}_\Gamma(\sigma) = \emptyset$  so indeed  $\Sigma$  is to be of form  $\mathcal{C}_\Gamma(\sigma) \cup \Phi$  where  $\Phi$  is a (possibly empty) subset of  $\mathcal{K}_\Gamma(\sigma)$ . Moreover, we wanted to ensure that  $\Sigma'$  is of the form  $\Sigma' = \mathcal{C}_\Delta(\sigma') \cup \Phi'$  where  $\Phi' \subseteq \mathcal{K}_\Delta(\sigma')$ . Referring back to the definitions of these sets in (59) and (59), we see that  $\mathcal{C}_\Delta(\sigma') = \{1, 2, 4\}$  so we satisfy the necessary condition. (For the record,  $\mathcal{K}_\Delta(\sigma') = \emptyset$  in this case.) Finally, no  $\Gamma$ -swaps or  $\Delta$ -swaps are possible so  $(\sigma, \Sigma)$  is a  $\Gamma$ -origin and  $(\sigma', \Sigma')$  is a  $\Delta$ -origin.

2. If  $\eta = (\square, \square, \circ, \circ, \square, \square, \circ, \square, \circ)$ ,

$$\begin{aligned} \psi((\square, *, \circ, \circ, \square, \square, *, *, \circ), \{1, 4, 5, 6, 7\}) \mapsto \\ ((\square, *, \circ, \circ, \square, \square, *, *, \circ), \{1, 2, 5, 6, 7\}) \end{aligned}$$

Note that there is no change in the signature from  $\sigma$  to  $\sigma'$  as no  $\square$ 's can move left in the strings of  $\square$ 's contained in  $\eta$ , and reflection in strings of  $\circ$ 's leaves these strings unchanged. The index sets are more interesting. From the definitions, we compute that  $\mathcal{C}_\Gamma(\sigma) = \{1, 4, 6, 7\}$ ,  $\mathcal{K}_\Gamma(\sigma) = \{5\}$ ,  $\mathcal{C}_\Delta(\sigma') = \{2, 7\}$ , and  $\mathcal{K}_\Delta(\sigma') = \{1, 5, 6\}$ . This illustrates that these sets may have very different cardinalities. We see that the sets  $\Sigma$  and  $\Sigma'$  are admissible.

3. If  $\eta = (\circ, \square, \circ, \square, \square, \square, \square, \circ)$ ,

$$\psi((\circ, \square, \circ, *, \square, \square, \square, *), \{3, 6, 7\}) \mapsto ((\circ, \square, \circ, \square, \square, *, \square, *), \{2, 6, 7\})$$

Here, the blocks of circles are all of length 1, so  $*$ 's and  $\circ$ 's in  $\sigma$  within these blocks do not change under  $\psi$  in  $\sigma'$ . We have  $\sigma_2 = \square$ , but  $2 \in \mathcal{N}_\Gamma(\sigma) \subseteq \Sigma$  so this  $\square$  remains fixed in  $\sigma'$ . In the block of 4  $\square$ 's, we see  $\sigma$  contains 3  $\square$ 's. The smallest index from this string which is in  $\Sigma$  is 6, corresponding to the last  $\square$ . So the first two  $\square$ 's move left, and the third remains fixed. Again, from the definitions, we compute that  $\mathcal{C}_\Gamma(\sigma) = \{3, 7\}$ ,  $\mathcal{K}_\Gamma(\sigma) = \{4, 5, 6\}$ ,  $\mathcal{C}_\Delta(\sigma') = \{2, 6\}$ , and  $\mathcal{K}_\Delta(\sigma') = \{4, 5, 7\}$ , so  $\Sigma$  and  $\Sigma'$  are admissible.

**Proof** The first thing to check is that if  $j \in \Sigma$  we may find  $v$  such that  $(j, v)$  are equalized. (If  $j \notin \Sigma$  this may not be true.) There may be several possible  $v$ 's, if  $\sigma'$  has  $\circ$ 's in the vicinity, and  $t$  will be the smallest. So the existence of  $v$  is all that needs to be proved – the condition that  $\sigma'_i \neq \circ$  has the effect of selecting the smallest, so that  $t$  will be uniquely determined.

If  $B$  is a  $\circ$ -block, then the existence of  $v$  is guaranteed by Lemma 11. If  $B$  is a  $\square$ -block, then  $B$  can be broken into segments in which  $\sigma$  and  $\sigma'$  are as follows. There is an initial segment (possibly empty) of  $\square$ 's that is common to both  $\sigma$  and  $\sigma'$ , and the remaining segments look like this:

$$\left\{ \begin{array}{cccccc} \sigma_l & \sigma_{l+1} & \sigma_{l+1} & \cdots & \sigma_{m-1} & \sigma_m \\ * & \square & \square & \cdots & \square & \square \end{array} \right\} \xrightarrow{\psi} \left\{ \begin{array}{cccccc} \sigma'_l & \sigma'_{l+1} & \cdots & \sigma'_{m-2} & \sigma'_{m-1} & \sigma'_m \\ \square & \square & \cdots & \square & \square & * \end{array} \right\}.$$

We claim that the only possible element of  $\Sigma$  in  $\{l, l+1, \dots, m\}$  is  $l$ . The reason is that if there was an element  $i$  of  $\Sigma$  in the range  $\{l+1, \dots, m\}$  the prescription for  $\sigma'$  would move the  $*$  to  $i-1$ , and this is not the case. Now  $(m+1, m+1)$  are equalized (or  $m=d$ ) and it follows that  $(l, l)$  are equalized. So we have the case  $j=l$ , and then we can take  $v=l$  also. It is easy to see that if  $j$  lies in the initial segment (if nonempty) that consists entirely of  $\square$ 's that we may take  $u=j$  in this case also.

This proves that  $t$  satisfying (63) exists.

We will make use of the following observation.

$$\text{If } \eta_{l'-1}\eta_{l'} = \circ\square \text{ and } \sigma'_{j'-1} = * \text{ for some } j' \leq l' \text{ then } l' \in \Sigma \text{ and } t(l') = l'. \quad (64)$$

To prove this, note that if  $\sigma_{l'} = *$  or if  $\sigma_{l'} = \square$  and  $\sigma_{l'-1} = \circ$  then  $l' \in \mathcal{C}_\Gamma(\sigma) \subseteq \Sigma$ . The fact that  $\sigma_i \neq \circ$  for some  $i < l'$ , needed here for the definition of  $\mathcal{C}_\Gamma(\sigma)$ , may be deduced from the fact that  $\sigma'_{j'-1} \neq \circ$  since it means that the  $\circ$ -block containing  $j'$  either is not the first block, or else contains some  $*$ 's for  $\sigma'$  and hence also for  $\sigma$ . Since  $\eta_{l'-1}\eta_{l'} = \circ\square$  this leaves only the case  $\sigma_{l'-1}\sigma_{l'} = *\square$ , and in this case  $l' \in \Sigma$  follows from the fact that  $\sigma$  is a  $\Gamma$ -origin by Lemma 12. Now  $(l', l')$  is equalized by Lemma 11 (i), and so  $t(l') = l'$ . This proves (64).

Now we need to check that  $\Sigma' = \{j'_1, j'_2, \dots\}$  is  $\Delta$ -admissible, that is,  $\mathcal{C}_\Delta(\sigma') \subseteq \Sigma' \subseteq \mathcal{C}_\Delta(\sigma') \cup \mathcal{K}_\Delta(\sigma')$ . That  $\Sigma' \subseteq \mathcal{C}_\Delta(\sigma') \cup \mathcal{K}_\Delta(\sigma')$  is almost immediate, as the set  $\mathcal{C}_\Delta(\sigma') \cup \mathcal{K}_\Delta(\sigma')$  contains every index  $j'$  with  $\sigma'_{j'-1} = \square$  or  $*$ , so long as  $\sigma'_{i'} \neq \circ$  for some  $i' \geq j'$ . Since each element  $\psi(j) = j' \in \Sigma'$  with  $j \in \Sigma$  has  $\sigma'_{j'-1} = \square$  or  $*$ , we need only check that  $\sigma'_{i'} \neq \circ$  for some  $i' \geq j'$ . This is clear since  $j \in \mathcal{C}_\Gamma(\sigma) \cup \mathcal{K}_\Gamma(\sigma)$  so  $P_\sigma(j)$  or  $Q_\sigma(j)$  is positive, and hence  $P_{\sigma'}(t)$  or  $Q_{\sigma'}(t)$  is positive, and  $j' \leq t$ .

We next show that  $\Sigma'$  contains  $\mathcal{C}_\Delta(\sigma')$ . Thus to each  $j' \in \mathcal{C}_\Delta(\sigma')$  we must find  $j \in \Sigma$  such that  $\psi(j) = j'$ .

First assume that  $\sigma'_{j'} = \circ$ . By definition of  $\mathcal{C}_\Delta(\sigma')$  we have  $\sigma'_{j'-1} \neq \circ$ . Also by definition of  $\mathcal{C}_\Delta(\sigma')$  there will be some  $l' > j'$  such that  $\sigma'_{l'} \neq \circ$ . Let  $l'$  be the smallest such value. Suppose that  $\eta_{l'} = \circ$ . Then  $\sigma'_{l'} = *$ . Since  $l'$  is the smallest value  $> j'$  such that  $\sigma'_{l'} \neq \circ$  we have  $\sigma'_i = \circ$  and hence  $\eta'_i = \circ$  for  $j' \leq i < l'$  and so the entire range  $j' \leq i \leq l'$  is contained within the same  $\circ$ -block  $B$ . By Lemma 11 (iii) there exists  $j \in B$  such that  $\sigma_j = *$  and  $j, j'$  are equalized, and moreover,  $\sigma_i \neq \circ$  for some  $i < j$ . Thus  $j \in \mathcal{C}_\Gamma(\sigma)$  so  $j \in \Sigma$  and  $t(j) = j'$ , so  $\psi(j) = j'$  (because  $\sigma'_{j'-1} \neq \circ$ ). Thus we may assume that  $\eta_{l'} = \square$ . We note that  $\eta_{l'-1} = \circ$  since  $\sigma'_{l'-1} = \circ$ . Thus  $l' \in \Sigma'$  and  $t(l') = l'$  by (64). Since  $\sigma'_{j'} = \sigma'_{j'-1} = \dots = \sigma'_{l'-1} = \circ$  but  $\sigma'_{j'-1} \neq \circ$  we have  $\psi(l') = j'$ . This finishes the case  $\sigma'_{j'} = \circ$ .

Next suppose that  $\sigma'_{j'} \neq \circ$ . Then  $\sigma'_{j'-1} = *$  since  $j' \in \mathcal{C}_\Delta(\sigma')$ . If  $\eta_{j'} = \circ$  then  $\sigma'_{j'}$  must be  $*$ . The assumption that  $j' \in \mathcal{C}_\Delta(\sigma')$  then implies that  $\sigma'_{j'-1} = *$  also and so Lemma 11 (iii) implies that  $t(j) = j'$  for some  $j$  in the same  $\circ$ -block as  $j'$ , with  $j \in \mathcal{C}_\Gamma(\sigma) \subseteq \Sigma$ . Then since  $\sigma'_{j'-1} \neq \circ$  we have  $\psi(j) = j'$ . Thus we may assume that  $\eta_{j'} = \square$ . In this case we will show that  $j' \in \Sigma$  and  $\psi(j') = j'$ . If  $\eta_{j'-1}\eta_{j'} = \square\square$  then since  $\sigma'_{j'-1} = *$  it follows from the description of  $\sigma'$  in  $\square$ -blocks (see (62)) that  $j' \in \Sigma$ , and by Lemma 11 (iii),  $(j', j')$  is equalized, so  $t(j') = j'$  and so since  $\sigma'_{j'-1} = *$  it follows that  $\psi(j') = j'$ . On the other hand if  $\eta_{j'-1}\eta_{j'} = \circ\square$  then we still have  $j' \in \Sigma$  by (64), and since  $\sigma'_{j'-1} = *$  we have  $\psi(j') = j'$ . This completes the proof that  $\Sigma'$  contains  $\mathcal{C}_\Delta(\sigma')$ .

Now we know that  $\Sigma'$  is  $\Delta$ -admissible for  $\sigma'$ . Next, we show that  $(\sigma', \Sigma')$  is a  $\Delta$ -origin. We must show that if  $\eta_{j'-1}\eta_{j'} = \square\circ$  and  $\sigma'_{j'-1}\sigma'_{j'} = \square*$  then  $j' \in \Sigma'$ . It follows from Lemma 11 that there exists  $j$  in the same  $\circ$ -block as  $j'$  such that  $\sigma_j = *$  and  $j \in \mathcal{C}_\Gamma(\sigma)$ , and  $(j, j')$  are equalized. Then  $t(j) = j'$  and since  $\sigma_{j'-1} \neq \circ$  we have  $\psi(j) = j'$ . Thus  $j' \in \Sigma'$ .

Next we observe that  $(\sigma, \Sigma)$  and  $(\sigma', \Sigma')$  concur. To see this, observe first that if  $j \in \Sigma$  and  $j' = \psi(j) \in \Sigma'$ , then  $(j, j')$  is equalized. This implies that the two sets (60) have the same cardinality (with  $\tau = \sigma'$ ). Moreover, if  $j$  is in a  $\circ$ -block, then  $j'$  is in the same block, while if  $j$  is in a  $\square$ -block then  $j' = j$  with the exception that if  $j$  is the left-most element of a  $\square$ -block, then  $j'$  can lie in the  $\circ$ -block to the left. These considerations show that  $\eta_i = \circ$  when (61) is satisfied. Therefore  $\sigma$  and  $\sigma'$  concur.

We see that  $\psi$  maps  $\Gamma$ -origins to  $\Delta$ -origins. To establish that it is a bijection between  $\Gamma$ -origins and  $\Delta$ -origins, we first note show the map  $\psi$  from  $\Gamma$ -origins to  $\Delta$ -origins is injective. Indeed, we can reconstruct  $\sigma$  and  $\Sigma$  from  $\sigma'$  and  $\Sigma'$  as follows. On  $\circ$ -blocks, the reconstruction is straightforward – the signature is just reversed on each  $\Delta$ -block, and the elements of  $\Sigma$  within a  $\Delta$ -block are just the values where  $\sigma_j = *$ , except that if  $\sigma_i = \circ$  for all  $i < j$  then  $j$  is omitted from  $\sigma$ . On  $\square$ -blocks,

the reconstruction of  $\Sigma$  must precede the reconstruction of  $\sigma$ . It follows from the preceding discussion that on the intersection of  $\Sigma$  with a  $\square$ -block,  $\psi$  is the identity map except that if the very first element of the block is in  $\Sigma$ ,  $\psi$  can move it into the preceding  $\circ$ -block. Thus if  $j \in \{1, 2, \dots, d\}$  and  $\eta_j = \square$  we can tell if  $j$  is in  $\Sigma$  as follows. If  $j$  not the first element on its block then  $j \in \Sigma$  if and only if  $j \in \Sigma'$ . If  $j$  is the first element, then  $j \in \Sigma$  if and only if  $j \in \Sigma'$  or (from the definition of  $\mathcal{C}_\Gamma$ ) if  $\sigma_{j-1} = \circ -$  and we recall that the signature is already known on  $\circ$ -blocks. Once  $\Sigma$  is known on  $\square$ -blocks,  $\sigma$  can be reconstructed by reversing the process that gave us  $\sigma'$ .

Since the map  $\psi$  is injective, we need only check that the number of  $\Delta$ -origins equals the number of  $\Gamma$ -origins. We extend  $\psi$  to a larger set by including  $\eta$  as part of the data: let  $\Omega_\Gamma$  be the set of all triples  $(\eta, \sigma, \Sigma)$  such that  $\eta$  is a nodal signature,  $\sigma$  a subsignature, and  $\Sigma$  a  $\Gamma$ -origin for  $\sigma$ ; and similarly we define  $\Omega_\Delta$ . Then  $\psi$  gives an injection  $\Omega_\Gamma \longrightarrow \Omega_\Delta$ . It will follow that  $\psi$  is a bijection if we show that the two sets have the same cardinality. A naive bijection between the two sets can be exhibited as follows. Let  $(\eta, \sigma, \Sigma)$  be given. Define  $(\tilde{\eta}, \tilde{\sigma}, \tilde{\Sigma})$  by  $\tilde{\eta}_i = \eta_{d-i}$ ,  $\tilde{\sigma}_i = \sigma_{d-i}$ , and  $\tilde{\Sigma} = \{d+1-j | j \in \Sigma\}$ . Note that  $\eta$  and  $\sigma$  are reversed in the range from 0 to  $d$ , while  $\Sigma$  is reversed in the range from 1 to  $d$ . Then  $(\tilde{\sigma}, \tilde{\Sigma})$  is a  $\Delta$ -origin if and only if  $(\sigma, \Sigma)$  is a  $\Gamma$ -origin, and so  $|\Omega_\Gamma| = |\Omega_\Delta|$ .  $\square$

## 13 Proof of Statement G

In Section 11 we reduced the proof to Statement G, given at the end of that Section, and we now have the tools to prove it.

**Lemma 12** *The cardinality of each  $\Gamma$ -pack or  $\Delta$ -pack is a power of 2.*

**Proof** In a  $\Gamma$ -swap  $*\square$  is replaced by  $\circ*$  in the signature. Since both signatures are subsignatures of  $\eta$ , this means that  $\eta$  has  $\circ\square$  at this location. From this it is clear that if a  $\Gamma$ -swap is possible at  $i-1, i$  then no swap is possible at  $i-2, i-1$  or  $i, i+1$ , and so the swaps are independent. Thus the cardinality of the pack is a power of 2.  $\square$

Given an origin  $(\sigma, \Sigma)$  for a  $\Gamma$ -equivalence class, define  $p_\Gamma(\sigma, \Sigma) = k$  where  $2^k$  is the cardinality of the  $\Gamma$ -pack to which the representative  $(\sigma, \Sigma)$  belongs. We may similarly define  $p_\Delta$  for  $\Delta$ -packs.

**Proposition 21** *Let  $(\sigma, \Sigma)$  be an origin for a  $\Gamma$ -equivalence class. Then  $p_\Gamma(\sigma, \Sigma)$ ,*

as defined above, can be given explicitly by

$$p_\Gamma(\sigma, \Sigma) = |\{i \in \{1, 2, \dots, d\} \mid (\sigma_{i-1}, \sigma_i) = (\circ, *), \eta_i = \square\}|. \quad (65)$$

**Proof** Recall that elements of a  $\Gamma$ -pack differ by a series of  $\Gamma$ -swaps from  $(\tau, T)$  to  $(\sigma, \Sigma)$ , which change  $\tau_{i-1}, \tau_i = *, \square$  to  $\sigma_{i-1}, \sigma_i = \circ, *$  provided  $i \notin T$ . Hence  $p_\Gamma(\sigma, \Sigma)$  is clearly at most the number of indices satisfying the condition on the right-hand side of (65).

Given an origin  $(\sigma, \Sigma)$ , let  $\tau \subseteq \eta$  be any subsignature possessing an  $(*, \square)$  at  $(\tau_{i-1}, \tau_i)$  where  $\sigma$  has a  $(\circ, *)$  at  $(\sigma_{i-1}, \sigma_i)$ . Let  $T$  be the set of indices obtained from  $\Sigma$  by replacing each such  $i \in \Sigma$  by  $i - 1$  (and leaving all other indices unchanged). We claim that  $(\tau, T) \sim_\Gamma (\sigma, \Sigma)$ . By our discussion above, it suffices to show that  $i \notin T$ . Indeed, by assumption,  $(\sigma_i, \sigma_{i+1}) \neq (\circ, *)$  so  $i + 1$  is not changed to  $i$  from  $\Sigma$  to  $T$  according to our rule. Moreover,  $i \in \Sigma$  is sent to  $i - 1 \in T$ . Hence (65) follows.  $\square$

**Lemma 13** *Let  $E_\Gamma$  be a  $\Gamma$ -pack with origin  $(\sigma, \Sigma)$ . Let  $\Sigma = \mathcal{C}_\Gamma(\sigma) \cup \Phi$  with  $\Phi \subseteq \mathcal{K}_\Gamma(\sigma)$  and let  $x = |\Phi|$ . Then*

$$\begin{aligned} \sum_{(\sigma, \Sigma) \in E_\Gamma} (-1)^\epsilon V(a_\Gamma(\sigma) + x, a_\Gamma(\sigma) + k_\Gamma(\sigma)) &= \\ (-1)^\epsilon V(a_\Gamma(\sigma) + x + p_\Gamma(\sigma, \Sigma), a_\Gamma(\sigma) + k_\Gamma(\sigma) + p_\Gamma(\sigma, \Sigma)), \end{aligned}$$

where  $\epsilon = \epsilon(\sigma)$  is the number of  $\circ$  in  $\sigma$  and  $\epsilon$  is the number of  $\circ$  in  $\sigma$ .

**Proof** It is easy to see that a  $\Gamma$ -swap does not change  $a_\Gamma(\sigma)$ , while it decreases  $k_\Gamma(\sigma)$  by 1. Thus repeatedly applying the identity

$$V(a, b) - V(a, b + 1) = V(a + 1, b + 1)$$

gives this result.  $\square$

**Lemma 14** *Let  $E_\Delta$  be a  $\Delta$ -pack with origin  $(\sigma, \Sigma)$ . Let  $\Sigma = \mathcal{C}_\Delta(\sigma) \cup \Phi$  with  $\Phi \subseteq \mathcal{K}_\Gamma(\sigma)$  and let  $x = |\Phi|$ . Then*

$$\begin{aligned} \sum_{(\sigma, \Sigma) \in E_\Delta} (-1)^\epsilon V(a_\Delta(\sigma) + x, a_\Delta(\sigma) + k_\Delta(\sigma)) &= \\ (-1)^\epsilon V(a_\Delta(\sigma) + x + p_\Delta(\sigma, \Sigma), a_\Delta(\sigma) + k_\Delta(\sigma) + p_\Delta(\sigma, \Sigma)), \end{aligned}$$

where  $\epsilon = \epsilon(\sigma)$  is the number of  $\circ$  in  $\sigma$  and  $\epsilon$  is the number of  $\circ$  in  $\sigma$ .

**Proof** Similar to the proof of Lemma 13.  $\square$

**Theorem 8** Let  $\psi$  be the bijection on equivalence classes given above, let  $(\sigma, \Sigma)$  be a  $\Gamma$ -origin and let  $\psi(\sigma, \Sigma) = (\sigma', \Sigma')$  be the corresponding  $\Delta$ -origin. Write  $\Sigma = \mathcal{C}_\Gamma(\sigma) \cup \Phi$  with  $\Phi \subseteq \mathcal{K}_\Gamma(\sigma)$  and similarly,  $\Sigma' = \mathcal{C}_\Delta(\sigma') \cup \Phi'$  with  $\Phi' \subseteq \mathcal{K}_\Delta(\sigma')$ . Then

$$\begin{aligned} V(a_\Gamma(\sigma) + |\Phi| + p_\Gamma(\sigma, \Sigma), a_\Gamma(\sigma) + k_\Gamma(\sigma) + p_\Gamma(\sigma, \Sigma)) &= \\ V(a_\Delta(\sigma') + |\Phi'| + p_\Delta(\sigma', \Sigma'), a_\Delta(\sigma') + k_\Delta(\sigma') + p_\Delta(\sigma', \Sigma')). & \end{aligned} \quad (66)$$

**Proof** First we will prove the equality of the second components

$$a_\Gamma(\sigma) + k_\Gamma(\sigma) + p_\Gamma(\sigma, \Sigma) = a_\Delta(\sigma') + k_\Delta(\sigma') + p_\Delta(\sigma', \Sigma'). \quad (67)$$

Consider the left-hand side of (67). The quantities  $k_\Gamma(\sigma)$  and  $a_\Gamma(\sigma)$  are defined in Section 11 before Proposition 17, where the latter is further defined in terms of  $n_\Gamma(\sigma)$  and  $t_\Gamma(\sigma)$ . Hence, the left-hand side of (67) is given by

$$\begin{aligned} & 2(d - (t_\Gamma - n_\Gamma)) + \left\{ \begin{array}{cc} -1 & \sigma_0 = \circ \\ 0 & \sigma_0 = \square \\ 1 & \sigma_0 = * \end{array} \right\} + \left\{ \begin{array}{cc} 1 & \sigma_d = \circ \\ 0 & \sigma_d \neq \circ \end{array} \right\} \\ & + |\{i \in [1, d] \mid \sigma_i = \square, \sigma_{i-1} \neq \circ\}| + |\{i \in [1, d] \mid (\sigma_{i-1}, \sigma_i) = (\circ, *), \eta_i = \square\}|, \end{aligned}$$

where  $[a, b]$  denotes  $\{x \in \mathbb{Z} \mid a \leq x \leq b\}$ . Now

$$2d - 2t_\Gamma + \left\{ \begin{array}{cc} -1 & \sigma_0 = \circ \\ 0 & \sigma_0 = \square \\ 1 & \sigma_0 = * \end{array} \right\} = 2d + 1 - 2BC(\sigma) + \left\{ \begin{array}{cc} 1 & \sigma_0 = \square \\ 0 & \sigma_0 \neq \square \end{array} \right\}$$

where  $BC(\sigma) = |\{i \in [0, d] \mid \sigma_i \neq *\}|$  is the total number of boxes and circles in  $\sigma$ . Also the quantity  $2n_\Gamma$  contributes a 2 for each  $i \in [1, d]$  with  $(\sigma_{i-1}, \sigma_i) = (\circ, \square)$ . We may regard this 2 as contributing 1 for each  $\square$  preceded by a  $\circ$  and 1 for each  $\circ$  followed by a  $\square$ . From this it follows that

$$\begin{aligned} & 2n_\Gamma + |\{i \in [1, d] \mid \sigma_i = \square, \sigma_{i-1} \neq \circ\}| + |\{i \in [1, d] \mid (\sigma_{i-1}, \sigma_i) = (\circ, *), \eta_i = \square\}| \\ & = |\{i \in [1, d] \mid \sigma_i = \square\}| + |\{i \in [0, d-1] \mid \sigma_i = \circ, \eta_{i+1} = \square\}|. \end{aligned}$$

Combining terms, we see that the left hand side of (67) is the sum of the two terms

$$2d + 1 - 2BC(\sigma) + |\{i \in [0, d] \mid \sigma_i = \square\}| \quad (68)$$

and

$$|\{i \in [0, d-1] \mid \sigma_i = \circ, \eta_{i+1} = \square\}| + \begin{cases} 1 & \sigma_d = \circ \\ 0 & \sigma_d \neq \circ \end{cases}. \quad (69)$$

Similarly, the right hand side of (67) is the sum of the two terms

$$2d + 1 - 2BC(\sigma') + |\{i \in [0, d] \mid \sigma'_i = \square\}| \quad (70)$$

and

$$|\{i \in [1, d] \mid \sigma'_i = \circ, \eta_{i-1} = \square\}| + \begin{cases} 1 & \sigma'_0 = \circ \\ 0 & \sigma'_0 \neq \circ \end{cases}. \quad (71)$$

Now, since the map  $\psi$  preserves the number of boxes and the number of circles in the signature, we have

$$BC(\sigma) = BC(\sigma')$$

and

$$|\{i \in [0, d] \mid \sigma_i = \square\}| = |\{i \in [0, d] \mid \sigma'_i = \square\}|.$$

Hence the quantities (68) and (70) are equal. The quantity (69) counts the number of  $i \in [0, d]$  such that  $\sigma_i = \circ$ ,  $\eta_{i+1} \neq \circ$  (this includes the possibility that  $i = d$  and  $\eta_{i+1}$  is not defined). But  $\psi$  reflects the entries of  $\sigma$  lying over strings of  $\circ$ 's in  $\eta$ . After doing so, each such  $i$  reflects to a  $\circ$  in  $\sigma'$  that is preceded by a  $\square$  (or is initial) in  $\eta$ . These are exactly the indices counted by (71). Hence they are equal.

This completes the proof of (67). To finish the proof of the theorem, we must show that

$$a_\Gamma(\sigma) + |\Phi| + p_\Gamma(\sigma, \Sigma) = a_\Delta(\sigma') + |\Phi'| + p_\Delta(\sigma', \Sigma').$$

By the construction of the bijection  $\psi$ , we have

$$c_\Gamma(\sigma) + |\Phi| = c_\Delta(\sigma') + |\Phi'|,$$

since these count the number of divisibility conditions, and this number is necessarily constant when the bijection obtains. In view of (67), it thus suffices to establish

$$c_\Gamma(\sigma) + k_\Gamma(\sigma) = c_\Delta(\sigma') + k_\Delta(\sigma'). \quad (72)$$

The case  $\eta_i = \sigma_i = \circ$  for all  $0 \leq i \leq d$  is trivial and we exclude it henceforth.

The quantity  $c_\Gamma(\sigma) + k_\Gamma(\sigma)$  counts the number of  $i \in [1, d]$  such that  $\sigma_i = \square$  or  $\sigma_i = *$  but  $\sigma_0, \dots, \sigma_{i-1}$  are not all  $\circ$ . We claim that (excluding the trivial case above)

$$c_\Gamma(\sigma) + k_\Gamma(\sigma) = |\{i \in [0, d] \mid \sigma_i = \square \text{ or } \sigma_i = *\}| - 1. \quad (73)$$

To check this, there are two cases. First, suppose  $\sigma_0 = \square$  or  $\sigma_0 = *$ . Then the index 0 is counted in the first term on the right hand side of (73) even though it is not in the range  $1 \leq i \leq d$ , but this is accounted for by subtracting 1 there. The indices  $i \in [1, d]$  with  $\sigma_i = \square$  or  $\sigma_i = *$  are counted on both sides. Hence (73) holds. The other possibility is  $\eta_0 = \sigma_0 = \circ$ . The index  $i = 0$  is not counted in the first term on the right hand side of (73). However,  $\sigma$  begins with a  $\circ$ , and the first index  $i_0$  such that  $\sigma_{i_0} \neq \circ$  is counted in the first term on the right hand side of (73). Subtracting 1 there makes up for the exclusion of the index  $i_0$  on the left hand side as it corresponds to a  $\square$  or  $*$  preceded by a nonempty initial string of  $\circ$ 's. The remaining indices  $i > i_0$  such that  $\sigma_i = \square$  or  $\sigma_i = *$  are counted on both sides. Hence (73) is also true in this case.

Similarly, we have (again excluding the case that all  $\sigma'_i = \circ$ )

$$c_{\Delta}(\sigma') + k_{\Delta}(\sigma') = |\{i \in [0, d] \mid \sigma'_i = \square \text{ or } \sigma'_i = *\}| - 1.$$

But since the map  $\psi$  preserves the number of boxes and the number of stars in the signature, we conclude that (72) holds, and the Theorem is proved.  $\square$

**Proof of Statement G:** Recall that Statement G was given at the conclusion of Section 10. On each side of that equality, we have partitioned the set of pairs  $(\sigma, \Sigma)$  in the sum into packs via their respective  $\Gamma$ - and  $\Delta$ -equivalence relations under  $\Gamma$ - and  $\Delta$ - swaps (as given in Definition 4), and then summed the total contribution of summands corresponding to each pack in Lemmas 13 and 14. (Note that to each member of a pack, the divisibility conditions appearing in the corresponding summands of Statement G are all simultaneously zero or simultaneously non-zero according to Lemma 19, i.e., they “concur” as in Definition 1.)

Moreover, we have established a bijection  $\psi$  between  $\Gamma$ -packs and  $\Delta$ -packs given as a map on origins in Proposition 20 with the further property that the origins (and hence all members of both packs) concur. Thus, according to the form of the equality given in Statement G, it suffices to verify two claims. First, that the statistics  $n_{\Gamma}(\sigma)$ ,  $\epsilon_{\Gamma}(\sigma)$  and their  $\Delta$  counterparts are preserved in both packs and under  $\psi$ ; this is easily verified from the definitions. The second more involved claim is that the quantities in Lemmas 13 and 14 are equal under the bijection  $\psi$ . But this is precisely the content of Theorem 8 and so Statement G is proved.  $\square$

This concludes the proof of the main theorem according to the chain of reductions A-G given in Section 2.

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