Orthogonality of Characters

The orthogonality of characters is a *part* of what is called Schur orthogonality. I gave a proof in class that is different from the proof in the book. Here it is. All vector spaces are over $\mathbb{C}$.

Let $\pi: G \longrightarrow \mathrm{GL}(V)$ be a representation of a finite group G. Its character is the function $\chi(g) = \mathrm{tr}(\pi(g))$. Define

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum \chi_1(g) \overline{\chi_2(g)}.$$

Lemma 1. *Let U and V be vector spaces, and let $A \in \mathrm{End}_{\mathbb{C}}(U)$, $B \in \mathrm{End}_{\mathbb{C}}(V)$. Let $W = \mathrm{Hom}_{\mathbb{C}}(U, V)$. Let $T: W \longrightarrow W$ be the linear transformation defined by $T(f) = B \circ f \circ A$. Then*

$$\mathrm{tr}(T) = \mathrm{tr}(A)\mathrm{tr}(B).$$

Proof. Let us choose bases of U and V so that A, B and f are represented by matrices (A_{ij}), (B_{ij}) and (f_{ij}). Then the matrix of Tf is (τ_{ij}) where

$$\tau_{ij} = \sum_{k,l} B_{ik} f_{kl} A_{li} = \sum_{k,l} C_{ij}^{kl} f_{kl}, \qquad C_{ij}^{kl} = B_{ik} A_{li}$$

Thus C_{ij}^{kl} are the entries in the matrix of a linear transformation representation representing T. The trace is the sum of the diagonal entries:

$$\mathrm{tr}(T) = \sum_{i,j} C_{ij}^{ij} = \sum_{i,j} B_{ii} A_{jj} = \mathrm{tr}(A) \cdot \mathrm{tr}(B). \qquad \square$$

A linear transformation $p: V \longrightarrow V$ is called a *projection* if it is idempotent, that is, $p^2 = p$.

Lemma 2. *Let $p: W \longrightarrow W$ be a projection. Then $W = \ker(p) \oplus \mathrm{im}(p)$. Moreover the trace of p is the dimension of the image of p.*

Proof. Please confirm that $W = \ker(p) \oplus \mathrm{im}(p)$. Choose a basis $x_1, \cdots, x_d$ of $\mathrm{im}(p)$. Complete it to a basis of W by adjoining a basis $y_1, \cdots, y_m$ of $\ker(p)$. With respect to the basis $x_1, \cdots, x_d, y_1, \cdots, y_m$ the matrix of p is

$$\begin{pmatrix} I_d & \\ & 0_m \end{pmatrix}$$

so $\mathrm{tr}(p) = d = \dim \mathrm{im}(p)$. $\square$

Theorem 3. *If χ_1 and χ_2 are the characters of representations π_1 and π_2 on V_1, V_2.*

$$\langle \chi_1, \chi_2 \rangle = \dim \mathrm{Hom}_G(V_1, V_2).$$

It is understood that $\mathrm{Hom}_G(V_1, V_2)$ is the space of G-module (i.e. $\mathbb{C}[G]$-module) homomorphisms. Since V_i is a $\mathbb{C}[G]$-module by means of $g\cdot v = \pi_i(g)v$ for $g\in G$ (extended to $\mathbb{C}[G]$ by linearity) this means that $\pi_2(g)\circ f = f\circ \pi_1(g)$ for $g\in G$ is the condition for $f\in \mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$ to be in $\mathrm{Hom}_G(V_1, V_2)$.

Proof. Let $W = \mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$. Define $p\colon W \longrightarrow W$ by

$$p(f) = \frac{1}{|G|}\sum_{g\in G} \pi_2(g)\circ f\circ \pi_1(g^{-1}). \tag{1}$$

We claim that p is a projection with image $\mathrm{Hom}_G(V_1, V_2)$. To see this it is sufficient to show that:

- If $f\in \mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$ then $pf\in \mathrm{Hom}_G(V_1, V_2)$,
- $f\in \mathrm{Hom}_G(V_1, V_2)$ then $pf = f$.

For the first assertion,

$$p(f)\circ \pi(h) = \frac{1}{|G|}\sum_{g\in G} \pi_2(g)\circ f\circ \pi_1(g^{-1}h).$$

Now the variable change $g\longmapsto hg$ shows that this equals

$$p(f)\circ \pi(h) = \frac{1}{|G|}\sum_{g\in G} \pi_2(h)\pi_2(g)\circ f\circ \pi_1(g^{-1}) = \pi_2(h)\circ p(f).$$

For the second, if $f\in \mathrm{Hom}_G(V_1, V_2)$ then each term in the sum on the right-hand side of (1) equals f and so $p(f) = f$.

Now by the Lemmas,

$$\dim \mathrm{Hom}_G(V_1, V_2) = \mathrm{tr}(p) = \frac{1}{|G|}\sum \mathrm{tr}\,\pi_2(g)\cdot \mathrm{tr}\,\pi_1(g^{-1}) = \frac{1}{|G|}\sum \chi_2(g)\cdot \overline{\chi_1(g)}.$$

This prove that $\dim \mathrm{Hom}_G(V_1, V_2) = \langle \chi_2, \chi_1\rangle = \overline{\langle \chi_1, \chi_2\rangle}$. But since this dimension is real, we may discard the complex conjugation. $\square$

Theorem 4. (Schur) *If χ_1 and χ_2 are the characters of irreducible representations π_1 and π_2. Then*

$$\langle \chi_1, \chi_2\rangle = \begin{cases} 1 & \text{if } \pi_1 \text{ and } \pi_2 \text{ are equivalent,}\\ 0 & \text{otherwise.}\end{cases}$$

Proof. Since now π_1 and π_2 are irreducible, Schur's Lemma asserts that

$$\dim \mathrm{Hom}_G(V_1, V_2) = \begin{cases} 1 & \text{if } V_1\cong V_2,\\ 0 & \text{otherwise,}\end{cases}$$

and the statement follow. $\square$