

Characters of Finite Abelian Groups

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These notes correspond to the optional lecture of April 1, which has the purpose of clarifying the relationship between group representation theory and Fourier analysis. This overlaps a bit with Section I.9 in Lang, who denotes the dual group G^\vee . I prefer G^* .

Note: After the optional lecture of April 2, the lectures will start with Chapter XVII of Lang's *Algebra*.

For April 8, do the exercises below.

Now let G be a finite abelian group, which we will write multiplicatively. Let $L^2(G)$ be the inner product space of all complex-valued functions on G , with the inner product

$$\langle \phi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \phi(g) \overline{\psi(g)}.$$

It is a finite-dimensional Hilbert space.

By a *linear character* χ of G , we mean a homomorphism $\chi : G \longrightarrow \mathbb{C}^\times$. The linear characters form an abelian group under multiplication, which we will denote G^* . If n is a positive integer, we will denote by μ_n the group of n -th roots of unity in \mathbb{C} .

Lemma 1 *If χ is a linear character of the finite group G , then $|\chi(g)| = 1$ for all $g \in G$. In fact, if $\chi(G) \subset \mu_n$, the group of n -th roots of unity in \mathbb{C} , where n is the exponent of G .*

Proof It is trivial that if $g \in G$ then $g^n = 1$, and so $\chi(g)^n = 1$, and so $\chi(g) \in \mu_n$. Of course this implies $|\chi(g)| = 1$ in \mathbb{C} . \square

Lemma 2 *If $\chi, \theta \in G^*$, then*

$$\langle \chi, \theta \rangle = \begin{cases} 1 & \text{if } \chi = \theta, \\ 0 & \text{otherwise.} \end{cases}$$

Proof If $\chi \neq \theta$ then $\chi(x) \neq \theta(x)$ for some $x \in G$. Since $|\theta(g)| = 1$, $\overline{\theta(g)} = \theta(g)^{-1}$ for all $g \in G$. Now

$$\langle \chi, \theta \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\theta(g)} = \frac{1}{|G|} \sum_{g \in G} \chi(g) \theta(g)^{-1}.$$

Now we permute the elements of G by making the substitution $g \mapsto gx$ and obtain

$$\langle \chi, \theta \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(xg) \theta(xg)^{-1} = \chi(x) \theta(x)^{-1} \frac{1}{|G|} \sum_{g \in G} \chi(xg) \theta(xg)^{-1} = \chi(x) \theta(x)^{-1} \langle \chi, \theta \rangle.$$

Since $\chi(x) \neq \theta(x)$, this implies that $\langle \chi, \theta \rangle = 0$.

On the other hand if $\chi = \theta$ then $\chi(g) = \theta(g)$ for all g , so

$$\langle \chi, \theta \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \theta(g)^{-1} = \frac{1}{|G|} \sum_{g \in G} 1 = 1.$$

□

We see that the linear characters of G are orthonormal. We will eventually show that they are an orthonormal basis of $L^2(G)$, but we need some further preparations before we can show this.

Lemma 3 *If G is finite abelian group and H a proper subgroup, and if χ is a linear character of H , then χ can be extended to a subgroup of G that is larger than H .*

To say that χ can be extended to a subgroup K of G that contains H means that we can find a linear character $\tilde{\chi}$ of K such that $\tilde{\chi}(h) = \chi(h)$ when $h \in H$.

Proof Let $x \in G - H$. There is a smallest positive integer d such that $x^d \in H$. Then $x^n \in H$ if and only if n is a multiple of d . Find a complex number a such that $a^d = \chi(x^d)$. Let $K = \langle H, x \rangle$. This is the subgroup of elements of G that can be written in the form $x^n h$ for some $h \in H$. We claim that we can define a character $\tilde{\chi}$ of K by $\tilde{\chi}(x^n h) = a^n \chi(h)$.

We must check that this is well defined. If $x^n h = x^m h'$, then $h' h^{-1} = x^{n-m}$, so $n - m$ is a multiple of d , say $n - m = dk$. Then

$$\chi(h') \chi(h)^{-1} = \chi(x^{n-m}) = \chi(x^d)^k = a^{dk} = a^{n-m},$$

which implies that $a^n \chi(h) = a^m \chi(h')$. Thus $\tilde{\chi}$ is well-defined. It is easily seen to be a homomorphism, that is, a linear character. □

Proposition 1 *Let G be a finite abelian group, and let H be a subgroup of G . Let χ be a linear character of H . Then χ can be extended to a linear character of G .*

Proof Let Σ be the set of subgroups K of G such that $K \supseteq H$ and χ can be extended to K . The set Σ is nonempty since $H \in \Sigma$, so let K be a maximal element. If K is a proper subgroup of G , then an extension of χ to K exists but cannot be extended to any larger subgroup, which contradicts Lemma 3. Thus $K = G$. \square

Proposition 2 *Let G be a finite abelian group, and let $x, y \in G$. If $\chi(x) = \chi(y)$ for all $\chi \in G^*$, then $x = y$.*

Proof Let $z = xy^{-1}$ have order n . We can define a linear character of $\langle z \rangle$ by $\chi(z^k) = e^{2\pi i k/n}$. If $z \neq 1$ then $\chi(z) \neq 1$, and extending χ to a character of G by Proposition 1 gives a contradiction. So $z = 1$ and $x = y$. \square

Let G be a finite abelian group, and let $x \in G$. Then x determines a function \check{x} on G^* , namely the map $\check{x}(\chi) = \chi(x)$. The fact that x is determined by \check{x} is a consequence of Proposition 2.

Proposition 3 *Let G be a finite abelian group.*

(i) *We have $|G| = |G^*|$.*

(ii) *If $x \in G$, then $\check{x} \in (G^*)^*$, and the map $x \mapsto \check{x}$ is an isomorphism $G \rightarrow (G^*)^*$.*

Proof We have $\check{x}(\chi\chi') = \chi\chi'(x) = \chi(x)\chi'(x) = \check{x}(\chi)\check{x}(\chi')$, so \check{x} is a character of G^* . To see that $x \mapsto \check{x}$ is a homomorphism, observe that

$$\check{x}\check{y}(\chi) = \check{x}(\chi)\check{y}(\chi) = \chi(x)\chi(y) = \chi(xy) = (\check{xy})(\chi),$$

because χ is a character. We see that $x \mapsto \check{x}$ is a homomorphism $G \rightarrow (G^*)^*$.

We can now prove (i) and (ii) simultaneously. We first observe that $|G^*| \leq |G|$ since the linear characters are an orthonormal set, hence linearly independent. Applying this twice, $|(G^*)^*| \leq |G|$. But $x \mapsto \check{x}$ is a homomorphism $G \rightarrow (G^*)^*$ that is injective by Proposition 2. We see that $|G| = |(G^*)^*|$ and $x \mapsto \check{x}$ is an isomorphism. Now $|G| = |(G^*)^*| \leq |G^*|$ and so $|G| = |G^*|$. \square

Because $x \mapsto \check{x}$ is an isomorphism, we may identify x with \check{x} and regard elements of G as characters of G^* . This means that the roles of G and G^* are symmetrical.

Theorem 1 *Let G be a finite abelian group. Then G^* is an orthonormal basis of $L^2(G)$.*

Proof We have already shown that G^* is an orthonormal set, hence linearly independent. But $|G^*| = |G| = \dim(L^2(G))$, and so they are a basis. \square

Exercise 1 If G and H are finite abelian groups, prove that

$$(G \times H)^* \cong G^* \times H^*.$$

Exercise 2 If G is a finite abelian group, prove that $G \cong G^*$. (**Hint:** reduce to the case of a cyclic group.)

Exercise 3 (Fourier inversion formula) Let $\mathcal{F} : L^2(G) \rightarrow L^2(G^*)$ be the *Fourier transform*, defined by $\mathcal{F}f = \hat{f}$, where \hat{f} is the function on $L^2(G^*)$ defined by

$$\hat{f}(\chi) = \frac{1}{\sqrt{|G|}} \sum_{x \in G} \chi(x) f(x).$$

Prove that

$$f(g) = \frac{1}{\sqrt{|G|}} \sum_{\chi \in G^*} \overline{\hat{f}(\chi)} \chi(g).$$

Exercise 4 (Plancherel formula) Prove that \mathcal{F} is an isometry, that is $\langle f_1, f_2 \rangle = \langle \hat{f}_1, \hat{f}_2 \rangle$.

Although Exercise 2.3 shows that $G \cong G^*$, this is a less natural isomorphism than the isomorphism $G \cong (G^*)^*$. The isomorphism $G \rightarrow (G^*)^*$ was defined in a canonical way, but any description of the isomorphism $G \cong G^*$ will depend on arbitrary choices. For example, if you solve Exercise 2 by first decomposing G as a direct product of cyclic groups, the proof will depend on the choice of this decomposition.

The operation $*$ is actually a *functor*, which means that it is not only an operation on abelian groups, but also on their homomorphisms. Indeed, if $f : G \rightarrow H$ is a homomorphism of abelian groups, then there is induced a homomorphism $f^* : H^* \rightarrow G^*$, which is composition with f . Thus if $\chi \in H^*$, then $\chi \circ f \in G^*$, and this is $f^*(\chi)$. Now the functor $*$ is *contravariant* since it reverses the direction of arrows. On the other hand, iterating it gives a *covariant* functor $**$, since the direction of arrows is twice reversed: since f^* is a map $H^* \rightarrow G^*$, $(f^*)^*$ is a map $(G^*)^* \rightarrow (H^*)^*$. Now the naturality of the isomorphism $\sim : G \rightarrow (G^*)^*$ can be

expressed with the observation that the following diagram commutes:

$$\begin{array}{ccc}
 G & \xrightarrow{f} & H \\
 \downarrow \wr & & \downarrow \wr \\
 (G^*)^* & \xrightarrow{(f^*)^*} & H
 \end{array}$$

No such property exists for the contravariant functor $*$.

Yet another reason that the isomorphism $G \cong G^*$ should be regarded as less fundamental than the isomorphism $G \cong (G^*)^*$ is that the whole theory can be generalized to the setting of topological groups. Specifically, let G be a *locally compact abelian group*. This means first of all, that G is a Hausdorff topological group (so that it is a Hausdorff topological space, and the group operations are continuous) and that every point has a neighborhood whose closure is compact; and that G is abelian. In this setting, everything we have done *except* Exercise 2.3 goes through without essential change. The characters $\chi : G \rightarrow \mathbb{C}^\times$ are required to be continuous and *unitary*, which means that $|\chi(g)| = 1$. The character group G^* is given the topology where a sequence converges if it converges uniformly on compact sets. We have $G \cong (G^*)^*$ (Pontriagin duality) and the Fourier transform is an isometry $L^2(G) \rightarrow L^2(G^*)$. Fourier analysis was first carried out in the setting of locally compact abelian groups in a monograph of André Weil.

However G and G^* may or may not be isomorphic. We have seen that they are isomorphic if G is finite; or if $G = \mathbb{R}$ (the additive group) or \mathbb{Q}_p (the additive group of p -adic numbers) then $G \cong G^*$. But if $G = \mathbb{R}/\mathbb{Z}$ then $G^* = \mathbb{Z}$, and it is in this setting that most people first encounter Fourier analysis. A function f on the circle $G = \mathbb{R}/\mathbb{Z}$ is transformed into a sequence of coefficients $\hat{f}(n)$ where

$$\hat{f}(n) = \int_0^1 f(x) e^{inx} dx.$$

The integer n corresponds to the character e^{inx} of \mathbb{R}/\mathbb{Z} , and the Plancherel formula is the assertion that

$$\int_0^1 f(x) \overline{f'(x)} dx = \sum_n \hat{f}(n) \overline{\hat{f}'(n)}.$$