Problems for May 15

Last week I assigned Problem 15 in Lang's algebra. I did not sufficiently make clear that I would like you to prove this *independently* of the theorems in Section 7 of Chapter 18 in Lang. Rather give a proof that is in the spirit of the notes on Mackey theory that are posted for this week.

Exercise 1. Let G be a group and V, W vector spaces. Let $C(G, V)$ be the space of all maps $G \longrightarrow V$. There is a representation $\rho_V: G \longrightarrow \mathrm{GL}(C(G, V))$ by right translation: for $f \in C(G, V)$ let $(\rho_V(g) f)(x) = f(xg)$. Let $T: C(G, V) \longrightarrow C(G, W)$ be a linear map that commutes with ρ, that is, $T \circ \rho_V(g) = \rho_W(g) \circ T$ for all $g \in G$. Prove that there is a map $\lambda: G \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V, W)$ such that T is convolution with λ. That is, $Tf = \lambda * f$ where

$$(\lambda * f)(x) = \sum_{g \in G} \lambda(g) f(g^{-1} x).$$

There is a hint for this problem. Ask or email me if you want to see it. Note that this is similar to Lemma 1 in the notes Mackey Theory posted on the class web page.

Exercise 2. If you used results from Section 7 of Lang to prove Exercise 15 from Lang, do it again without using them. Instead use Exercise 1.

If you did Exercise 1 for this week in the course of doing Exercise 15, you don't have to do Exercise 1. If you did Exercise 15 without using results from Section 7 of Lang, you don't have to do Exercise 2.

Exercise 3. Explain how Exercise 15 (which we have seen can be proved directly) implies Theorem 7.7 on page 695.

Exercise 4. Explain how Theorem 7.7 implies Theorem 7.6. (Use Frobenius reciprocity.)

The notion of *central character* is explained in the posted notes on $\mathrm{SL}(2, 3)$.

Exercise 5. (Finite Stone-Von Neumann Theorem) Let H be the "Heisenberg" group of order q^3 (where q is a prime power) consisting of

$$\left\{ \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} | x, y, z \in \mathbb{F}_q \right\}.$$

(a) Identify the center Z of H and explain why any subgroup of H that contains Z is normal. Observe that H/Z is a vector space of dimension 2 over $\mathbb{F}_q$ and deduce that there are $q + 1$ subgroups of H of order q^2 that contain Z.

(b) Let A and B be two of these subgroups, and let ψ_A, ψ_B be linear characters that are nontrivial when restricted to Z. Use Mackey theory to prove that

$$\dim \mathrm{Hom}_G(\psi_A^G, \psi_B^G) = \begin{cases} 1 & \text{if } \psi_A \text{ and } \psi_B \text{ have the same restriction to } Z \\ 0 & \text{otherwise.} \end{cases}$$

(c) Deduce that ψ_A^G is irreducible if ψ_A restricted to Z is nontrivial.

(d) Show that if ω is any nontrivial linear character of Z, then H has a unique irreducible representation with central character ω.