Math 210C: Second Midterm

Hecke Algebras. The ideas in this exam all have analogs in the theory of representations of p-adic groups. They are important in the theory of automorphic forms.

Let G be a finite group and let H_1, H_2 and H_3 be subgroups. Let ψ_1, ψ_2, ψ_3 be characters of H_1, H_2 and H_3 respectively.

Exercise 1. (a) Remind me why $\mathrm{Hom}_G(\psi_1^G, \psi_2^G)$ is isomorphic to the space of $\Delta: G \longrightarrow \mathbb{C}$ such that $\Delta(h_2 g h_1) = \psi_2(h_2)\Delta(g)\psi_1(h_1)$. (Just state the isomorphism. Don't prove anything.)

(b) If $T: \psi_1^G \longrightarrow \psi_2^G$ and $U: \psi_2^G \longrightarrow \psi_3^G$ are homomorphisms corresponding to functions Δ_T and Δ_U show that $U \circ T: \psi_1^G \longrightarrow \psi_3^G$ corresponds to some constant times $\Delta_U * \Delta_T$ where the convolution is

$$(\Delta_U * \Delta_T)(g) = \sum_{x \in G} \Delta_U(gx)\Delta_T(x^{-1}).$$

If V is a G-module, note that $\mathrm{End}_G(V)$ is a ring. If $V = \oplus\, d_i V_i$ where V_i are distinct irreducibles, we say that V is *multiplicity free* if all d_i are $\leqslant 1$.

Exercise 2. Show that if V is a G-module, and $V = \oplus\, d_i V_i$ where V_i are distinct irreducibles, then $\mathrm{End}_G(V) = \prod \mathrm{Mat}_{d_i}(\mathbb{C})$. Deduce that V is multiplcity free if and only if $\mathrm{End}_G(V)$ is commutative.

Exercise 3. If ψ is a character of the subgroup H of G, show that ψ^G is multiplicity-free if and only if the convolution ring of $\Delta: G \longrightarrow \mathbb{C}$ that satisfy

$$\Delta(h_2 g h_1) = \psi(h_2)\Delta(g)\psi(h_1) \tag{1}$$

is a commutative ring.

Gelfand gave a method of proving that such *Hecke algebras* are commutative. Here is an example.

Exercise 4. (Gelfand-Graev) Let $G = \mathrm{GL}(2, \mathbb{F}_p)$ and let $H = \left\{ \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} | x \in \mathbb{F}_p \right\}$. It is a p-Sylow subgroup. Let $\psi_0: F \longrightarrow \mathbb{C}$ be a nontrivial additive character and define $\psi: H \longrightarrow \mathbb{C}$ by

$$\psi\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} = \psi_0(x).$$

Let $\mathcal{H}$ be the convolution ring of $\Delta: G \longrightarrow \mathbb{C}$ that satisfy (1).

(a) Show that if $\Delta \in \mathcal{H}$ then $\Delta(g) = 0$ unless g is in one of the double cosets

$$H\begin{pmatrix} a & \\ & a \end{pmatrix}H, \qquad H\begin{pmatrix} & b \\ c & \end{pmatrix}H.$$

Hint: Show that the set of double cosets is

$$H\begin{pmatrix} a & \\ & d \end{pmatrix}H, \qquad H\begin{pmatrix} & b \\ c & \end{pmatrix}H,$$

then show that in the first case $a = d$.

(b) Let $\theta: G \longrightarrow G$ be the involution

$$\theta(g) = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \cdot {}^t g \cdot \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}.$$

This is an *antiautomorphism* so $\theta(g_1 g_2) = \theta(g_2)\theta(g_1)$. Show that if $\theta(H) = H$ and that $\psi \circ \theta = \psi$. Deduce that if $\Delta \in \mathcal{H}$ then $\Delta^\theta \in \mathcal{H}$ where

$$\Delta^\theta(g) = \Delta(\theta(g))$$

then $\Delta \mapsto \Delta^\theta$ is an antiautomorphism of $\mathcal{H}$.

(c) Use (a) to show that $\Delta = \Delta^\theta$ and conclude that $\mathcal{H}$ is commutative.

Exercise 5. ("Uniqueness of Whittaker models") Let G, H, ψ be as in the last exercise. Let (π, V) be an irreducible G-module. Show that the space of functionals $\Lambda: V \longrightarrow \mathbb{C}$ that satisfy

$$\Lambda\left(\pi\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} v\right) = \psi_0(x)\Lambda(v)$$

is at most one dimensional.

Hint: Interpret Λ as an element of $\mathrm{Hom}_H(V, \psi)$ and use Frobenius reciprocity.