Mackey Theory

The contents of the lectures for May 4 and May 5, 2009.

1 Mackey Theory for linear characters

Mackey theory asks the following question: if H_1 and H_2 are subgroups of G and ψ_1 and ψ_2 are characters, then what is $\left\langle \psi_1^G, \psi_2^G \right\rangle$? This may seem like a technical question, and indeed many accounts of Mackey theory may not do much to dispel this impression. However Mackey's theorem is extremely important and useful, and properly understood it has a conceptual basis, which we hope to convey.

For simplicity, we will limit ourselves to the special case where ψ_1 and ψ_2 are linear characters, which makes for a minor simplification, and is already enough for some important examples.

We go back to the viewpoint that $\mathbb{C}[G]$ is the ring (under convolution) of complex valued functions on G. We recall the right regular representation $\rho: G \longrightarrow \mathrm{End}(\mathbb{C}[G])$, which is the action $(\rho(g) f)(x) = f(x g)$.

Lemma 1. *Let $T: \mathbb{C}[G] \longrightarrow \mathbb{C}[G]$ be a linear transformation that commutes with $\rho(g)$; that is, $T(\rho(g) f) = \rho(g) T(f)$. Then there exists a unique $\lambda \in \mathbb{C}[G]$ such that $T(f) = \lambda * f$.*

Proof. Define $\delta_0(g) = 1$ if $g = 1$, and 0 if $g \neq 1$. Then δ_0 is the unit in the convolution ring $\mathbb{C}[G]$, that is, $\delta_0 * f = f * \delta_0 = f$ for all $f \in \mathbb{C}[G]$. If λ exists such that $T(f) = \lambda * f$ for all f, then $\lambda = \lambda * \delta_0 = T(\delta_0)$. Hence it is unique, and it remains to be shown that $\lambda = T(\delta_0)$ works. We claim that any $f \in \mathbb{C}[G]$ can be written as

$$f = \sum_{g \in G} f(g) \rho(g^{-1}) \delta_0. \tag{1}$$

Indeed, applying the right-hand side to $x \in G$ gives

$$\sum_{g \in G} f(g) (\rho(g^{-1}) \delta_0)(x) = \sum_{g \in G} f(g) \delta_0(x g^{-1}).$$

Only one term contributes, which is $g = x$, and that term equals $f(x)$. This proves (1).

Now applying T to (1) gives

$$T f = \sum_{g \in G} f(g) T\big(\rho(g^{-1}) \delta_0\big) = \sum_{g \in G} f(g) \rho(g^{-1}) T(\delta_0) = \sum_{g \in G} f(g) \rho(g^{-1}) \lambda.$$

Thus

$$T f(x) = \sum_g (\rho(g^{-1}) \lambda)(x) f(g) = \sum_g \lambda(x g^{-1}) f(g) = (\lambda * f)(x). \qquad \square$$

We may regard ψ_i as the character of H_i acting on $V_i=\mathbb{C}$ (since V_i is one-dimensional) with the representation $\pi_i(g)v=\psi_i(g)v$ when $g\in G$ and $v\in V_i=\mathbb{C}$. Then ψ_i^G acts on the space V_i^G of all functions $f_i\colon G\longrightarrow\mathbb{C}$ $(=V_i)$ such that $f_i(h_ig)=\psi_i(h_i)f(g)$ when $h_i\in H_i$. The subspaces V_i^G are thus invariant subspaces of $\mathbb{C}[G]$ under the action ρ, and the action of G on V_i^G is given by ρ.

Theorem 2. (Geometric form of Mackey's Theorem) *Let* $\Lambda\in\mathrm{Hom}_G(V_1^G,\,V_2^G)$. *Then there exists a function* $\Delta\in\mathbb{C}[G]$ *such that*

$$\Delta(h_2gh_1)=\psi_2(h_2)\Delta(g)\psi_1(h_1),\qquad h_i\in H_i, \tag{2}$$

and $\Lambda f=\Delta*f$ *for all* $f\in\psi_1^G$. *The map* $T\longmapsto\Delta$ *is a vector space isomorphism of* $\mathrm{Hom}_G(\psi_1^G,\psi_2^G)$ *with the space of all functions satisfying (2).*

Proof. (sketch) If $f\in\mathbb{C}[G]$ define $p(f)$ to be the function

$$p(f)(g)=\frac{1}{|H_1|}\sum_{h\in H_1}\psi_1(h)f(h^{-1}g).$$

The following facts are easily verified (Exercise 1). First p is a projection operator with image V_1^G; this means that $p(f)\in V_1^G$, and that $p(f)=f$ if $f\in V_1^G$. Moreover, $p(\rho(g)f)=\rho(g)p(f)$. Now we define $T\colon\mathbb{C}[G]\longrightarrow\mathbb{C}[G]$ to be $\Delta\circ p$. Then since Λ is a G-module homomorphism, we have $\Lambda\circ\rho(g)=\rho(g)\circ\Lambda$, and so T satisfies $T\circ\rho(g)=\rho(g)\circ T$. Therefore by the Lemma we have $Tf=\Delta*f$ for some unique Δ. Let us check that Δ has the property (2). This can be separated into two statements,

$$\Delta(gh_1)=\Delta(g)\psi_1(h_1),\qquad h_1\in H_1, \tag{3}$$

and

$$\Delta(h_2g)=\psi_2(h_2)\Delta(g),\qquad h_2\in H_2. \tag{4}$$

We will prove (3) and leave (4) to the reader, with a hint. For (3) we note that

$$\Delta*p(f)=\Lambda\circ p\circ p(f)=\Lambda(p(f))=\Delta*f.$$

This means that

$$\sum_{y\in G}\Delta(y)f(y^{-1}g)=(\Delta*f)(g)=(\Delta*p(f))(g)\ =$$

$$\sum_{y\in G}\Delta(y)\frac{1}{|H_1|}\sum_{h\in H_1}\psi_1(h_1)f(h^{-1}y^{-1}g)$$

Interchanging the order of summation and (for fixed h_1) making the variable change $y\longmapsto yh_1^{-1}$ gives

$$\sum_{y\in G}\Delta(y)f(y^{-1}g)=\frac{1}{|H_1|}\sum_{h\in H_1}\sum_{y\in G}\Delta(yh^{-1})\psi_1(h)f(y^{-1}g).$$

Since this is true for all f, we can deduce that

$$\Delta(y)=\frac{1}{|H_1|}\sum_{h\in H_1}\Delta(yh^{-1})\psi_1(h).$$

From this identity (3) is easily deduced: we have

$$\Delta(yh_1) = \frac{1}{|H_1|}\sum_{h\in H_1} \Delta(yh_1\,h^{-1})\psi_1(h).$$

On making the variable change $h \longmapsto h h_1$ this turns into

$$\left(\frac{1}{|H_1|}\sum_{h\in H_1} \Delta(y\,h^{-1})\psi_1(h)\right)\psi_1(h_1) = \Delta(y)\,\psi_1(h_1).$$

We leave (4) to the reader, with the hint that it follows from the fact that the image of T is contained in V_2^G. □