

Second Take-Home Midterm

Frobenius-Schur Numbers

Rules: Due Wednesday, May 28 at 5 PM. I will be away during the week of May 19 but will have office hours on May 26, 27 and 28 as usual. You may work together but **write up the solutions individually**. In other words, discuss the problems all you want but don't show one another your written solutions. You may want to come to class during the week of May 19 even though I won't be here to discuss the midterm. You may use Lang's *Algebra* but no other sources.

The first exercise is related to the material of Chapter 19 but I'm suggesting a variation of his construction of $\wedge^k V$ which is equivalent in characteristic zero, and we'll also need the symmetric power $\vee^k V$ (often denoted $\text{Sym}^k(V)$).

Let V be a finite-dimensional vector space. Convince yourself that there are commuting actions of S_k and $\text{GL}(V)$ on $\otimes^k V = V \otimes \cdots \otimes V$, in which $\sigma \in S_k$ has the effect

$$\sigma(v_1 \otimes \cdots \otimes v_k) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}$$

and $g \in \text{GL}(V)$ has the effect

$$g(v_1 \otimes \cdots \otimes v_k) = gv_1 \otimes \cdots \otimes gv_k.$$

\mathfrak{a}_\vee be the subspace generated by tensors of the form $\sigma \cdot \xi - \xi$ for $\sigma \in S_k$ and $\xi \in \otimes^k V$. Define \mathfrak{a}_\wedge to be the subspace defined by tensors of the form $\sigma \cdot \xi - \text{sgn}(\sigma) \cdot \xi$. Define

$$\vee^k V = \otimes^k V / \mathfrak{a}_\vee, \quad \wedge^k V = \otimes^k V / \mathfrak{a}_\wedge.$$

They are called the *symmetric* and *exterior powers* of V . If v_1, \dots, v_k are in V , define $v_1 \vee \cdots \vee v_k$ and $v_1 \wedge \cdots \wedge v_k$ to be the images of $v_1 \otimes \cdots \otimes v_k$ in $\vee^k V$ and $\wedge^k V$ respectively.

Problem 1: Suppose that x_1, \dots, x_d is a basis of V . Show that

$$x_{i_1} \otimes \cdots \otimes x_{i_k} \quad (1 \leq i_1, \dots, i_k \leq d)$$

is a basis of $\otimes^k V$, that

$$x_{i_1} \vee \cdots \vee x_{i_k} \quad (1 \leq i_1 \leq \cdots \leq i_k \leq d)$$

is a basis of $\vee^k V$, and that

$$x_{i_1} \wedge \cdots \wedge x_{i_k} \quad (1 \leq i_1 < \cdots < i_k \leq d)$$

is a basis of $\wedge^k V$. What are the dimensions of $\otimes^k V$, $\vee^k V$ and $\wedge^k V$?

Problem 2: Show that \mathfrak{a}_\vee and \mathfrak{a}_\wedge are invariant under the action of $\text{GL}(V)$, and conclude that there are group homomorphisms $\vee^k : \text{GL}(V) \rightarrow \text{GL}(\vee^k V)$ and $\wedge^k : \text{GL}(V) \rightarrow \text{GL}(\wedge^k V)$ such that

$$\begin{aligned} (\vee^k g)(v_1 \vee \cdots \vee v_k) &= gv_1 \vee \cdots \vee gv_k, \\ (\wedge^k g)(v_1 \wedge \cdots \wedge v_k) &= gv_1 \wedge \cdots \wedge gv_k. \end{aligned}$$

Show that if $\alpha_1, \dots, \alpha_d$ are the eigenvalues of g then

$$\begin{aligned} \operatorname{tr} \vee^k g &= h_k(\alpha_1, \dots, \alpha_d), & \operatorname{tr} \wedge^k g &= e_k(\alpha_1, \dots, \alpha_d), \\ h_k(\alpha_1, \dots, \alpha_d) &= \sum_{1 \leq i_1 \leq \dots \leq i_k \leq d} \alpha_{i_1} \cdots \alpha_{i_k}, & e_k(\alpha_1, \dots, \alpha_d) &= \sum_{1 \leq i_1 < \dots < i_k \leq d} \alpha_{i_1} \cdots \alpha_{i_k}. \end{aligned}$$

If $k=2$ and $d=3$, write out h_k and e_k explicitly.

Hence if we have a representation $\pi: G \rightarrow \operatorname{GL}(V)$ we may compose these with these homomorphisms and get homomorphisms $\vee^k \pi: G \rightarrow \operatorname{GL}(\vee^k V)$ and $\wedge^k \pi: G \rightarrow \operatorname{GL}(\wedge^k V)$ called the *symmetric* and *exterior powers* of π . Let χ , $\vee^k \chi$ and $\wedge^k \chi$ be the characters of π , $\vee^k \pi$ and $\wedge^k \pi$, respectively.

Problem 3: Show that

$$\alpha_1^2 + \alpha_2^2 + \dots + \alpha_d^2 = h_2(\alpha_1, \dots, \alpha_d) - e_2(\alpha_1, \dots, \alpha_d),$$

and deduce that

$$\chi(g^2) = \vee^2 \chi - \wedge^2 \chi.$$

Problem 4: Assume that χ is irreducible. Show that χ is real-valued if and only if $\langle \chi^2, 1 \rangle = 1$. Show that $\chi^2 = \wedge^2 \chi + \vee^2 \chi$. Conclude that if χ is real valued then exactly one of the alternatives:

$$\langle \wedge^2 \chi, 1 \rangle = 1 \quad \text{and} \quad \langle \vee^2 \chi, 1 \rangle = 0,$$

or

$$\langle \wedge^2 \chi, 1 \rangle = 0 \quad \text{and} \quad \langle \vee^2 \chi, 1 \rangle = 1$$

is true. Deduce that

$$\frac{1}{|G|} \sum \chi(g^2) = 0$$

if χ is not real-valued, while

$$\frac{1}{|G|} \sum \chi(g^2) = \pm 1$$

if χ is real-valued.

Define the *Frobenius-Schur number* (when χ is irreducible)

$$\varepsilon_\chi = \frac{1}{|G|} \sum \chi(g)^2.$$

We see it has value 0, 1 or -1 . Roughly speaking, the Frobenius-Schur number detects whether the natural field of definition of the representation is \mathbb{C} , \mathbb{R} or \mathbb{H} . Indeed, there are natural homomorphisms $\operatorname{GL}(n, \mathbb{R}) \rightarrow \operatorname{GL}(n, \mathbb{C})$ and $\operatorname{GL}(n, \mathbb{H}) \rightarrow \operatorname{GL}(2n, \mathbb{C})$ and if a complex representation factors through one of these its FS number is 1 or -1 . You do not have to prove this fact. But if $\varepsilon_\chi = 1$ the representation is called *real* or *orthogonal*, while if $\varepsilon_\chi = -1$ it is called *quaternionic* or *symplectic*.

Problem 5: Compute ε_χ when χ is the degree 2 irreducible character of G for $G = D_8$ (the dihedral group of degree 8) or $G = Q$ (the quaternion group of degree 8) or when χ is either of the degree 3 irreducible characters of A_5 .

The aim of the next problem is to prove the following fact. Let $g, h \in G$. Then *column orthogonality* is the statement

$$\sum_{\chi} \chi(g) \overline{\chi(h)} = \begin{cases} |C_G(g)| & \text{if } g \text{ and } h \text{ are conjugate,} \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

Here $C_G(g)$ is the centralizer of g .

Problem 6: Let χ_i be the irreducible characters of G and let g_j be representatives of the conjugacy classes. Define $U = (u_{ij})$ to be the matrix whose entries are

$$u_{ij} = \frac{\chi_i(g_j)}{\sqrt{|C_G(g_j)|}}.$$

Show that U is a unitary matrix, that is, $U \cdot \overline{U^t} = I$. Remind us why the transpose of a unitary matrix is unitary, and deduce column orthogonality (1) from the fact that U^t is unitary.

Problem 7: Let G be a group and let $a \in G$. Evaluate the sum

$$\frac{1}{|G|} \sum_{g \in G} \sum_{\text{irreducible } \chi} \overline{\chi(a)} \cdot \chi(g^2)$$

in two different ways and deduce that the function $M = \sum \varepsilon_\chi \chi$ has the following combinatorial interpretation: $M(a)$ is the number of solutions to the equation $x^2 = a$ with x in G . Deduce from this that $\sum_{\chi} \chi(1)$ is the number of $x \in G$ such that $x^2 = 1$, then $\sum_{\chi} \chi(a)$ is the number of $x \in G$ such that $x^2 = a$ for all a .

Problem 8: Let $G = S_n$ verify by straightforward checking for $n \leq 6$ that the sum of the degrees of the irreducible characters is the number of $x \in G$ such that $x^2 = 1$. (This fact is true for all n but since you know the character degrees for $n \leq 6$ you can check it by hand.)