1 Next Homework

For May 14, do (in Chapter 18) numbers 4, 5, 7, 15. Here are some more specific instructions for number 15. Modify slightly the defining formula for functions f in $M_G(F_1, F_2)$: they are to satisfy

$$f(h_2 g h_1) = \rho_2(h_2) f(g) \rho_1(h_1).$$

Construct the isomorphism so that f corresponds to $L \in \mathrm{Hom}_G(F_1^G, F_2^G)$ where

$$L\phi = f * \phi \in F_2^G, \qquad \phi \in F_1^G,$$

we define the convolution as follows:

$$(f * \phi)(g) = \sum_{x \in G} f(gx) \rho_1(x^{-1}).$$

A Correction

Problem 4 is misstated in Lang. He means: Let S be a normal subgroup of G. Let ψ be an irreducible character of S. Then ψ^G is irreducible if and only if:

$$[\sigma]\psi = \psi \text{ for } \sigma \in G \text{ if and only if } \sigma \in S.$$

As in Section 7 and other problems, $([\sigma]\psi)(g) = \psi(\sigma^{-1} g \sigma)$.

2 Remarks on $\mathbf{SL_2(\mathbb{F}_3)}$

The group $\mathrm{SL}_2(\mathbb{F}_3)$ is a nonabelian group of order 24. It is somewhat related to the group S_4, which is another nonabelian group of order 24, in that we have short exact sequences:

$$1 \longrightarrow Z_2 \longrightarrow \mathrm{SL}_2(\mathbb{F}_3) \longrightarrow A_4 \longrightarrow 1,$$

$$1 \longrightarrow A_4 \longrightarrow S_4 \longrightarrow Z_2 \longrightarrow 1.$$

The groups are thus twins in that they have the same composition factors in the Jordan-Hölder Theorem. The homomorphism $\mathrm{SL}_2(\mathbb{F}_3) \longrightarrow A_4$ corresponds to the action on the 3-Sylow subgroups, or on the projective line $\mathbb{P}^1(\mathbb{F}_3) = \mathbb{F}_3 \cup \{\infty\}$.

However computing the character table of $\mathrm{SL}_2(\mathbb{F}_3)$ is harder than computing the character table of S_4. Here are the conjugacy classes. We give the order of each conjugacy class, and a representative.

1 $\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$	1 $\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$	6 $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$	4 $\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$	4 $\begin{pmatrix} 1 & -1 \\ & 1 \end{pmatrix}$	4 $\begin{pmatrix} -1 & 1 \\ & -1 \end{pmatrix}$	4 $\begin{pmatrix} -1 & -1 \\ & -1 \end{pmatrix}$

Using the homomorphism $\mathrm{SL}_2(\mathbb{F}_3) \longrightarrow A_4$, we can pull back the four irreducible characters of A_4 and obtain the following four characters. Here $\rho = e^{2\pi i/3}$.

	1 $\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$	1 $\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$	6 $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$	4 $\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$	4 $\begin{pmatrix} 1 & -1 \\ & 1 \end{pmatrix}$	4 $\begin{pmatrix} -1 & -1 \\ & -1 \end{pmatrix}$	4 $\begin{pmatrix} -1 & 1 \\ & -1 \end{pmatrix}$
χ_1	1	1	1	1	1	1	1
χ_2	1	1	1	ρ	ρ^2	ρ	ρ^2
χ_3	1	1	1	ρ^2	ρ	ρ^2	ρ
χ_4	3	3	-1	0	0	0	0

The remaining 3 characters are surprisingly hard to construct, and we will introduce some new concepts in order to get them. First, we note the concept of the *central character* of a representation.

Proposition 1. *Let (π, V) be an irreducible representation of the finite group G. Then there exists a linear character ω of the center $Z(G)$ such that if $z \in Z(G)$ then $\pi(z)$ is the scalar linear transformation $\omega(z) \cdot I$. (Here $I: V \longrightarrow V$ is the identity map.) So if $\chi = \chi_\pi$ we have $\chi(z) = d\chi(z)$ for all $z \in Z(G)$, where $d = \dim(V)$.*

Proof. Since $z \in Z(G)$ we have $zg = gz$ for all $g \in G$, so $\pi(z)\pi(g) = \pi(g)\pi(z)$. This means that $\pi(z) \in \mathrm{Hom}_{\mathbb{C}[G]}(V, V)$. So by Schur's Lemma (Lemma ? (ii)) there is a scalar $\omega(z)$ such that $\pi(z)v = \omega(z)v$ for all $v \in V$. We note that if $z, z' \in Z(G)$ and $0 \neq v \in V$ is any nonzero element we have

$$\omega(zz')v = \pi(zz')v = \pi(z)\pi(z')v = \omega(z)\omega(z')v,$$

so $\omega(zz') = \omega(z)\omega(z')$, and ω is a linear character of $Z(G)$. □

In the case at hand, $Z(G) = \{\pm I\}$ has order 2, and a glance at the characters we've constructed shows that they all have trivial central character. Our search for characters to induce can be narrowed with this in mind. We will seek subgroups $H \subset G$ such that H contains $Z(G)$, and the characters that we will induce will have to have nontrivial central characters.

Unfortunately, we won't be able to make do with a single subgroup H. We'll use two different ones, then take linear combinations of the characters that we induce in order to get an irreducible one. The first H that we will consider is a 2-Sylow subgroup. There are 3 such 2-Sylow subgroups, but let us take this one:

$$Q = \langle x, y \rangle, \qquad x = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

We have $x^2 = y^2 = -I$ and $yxy^{-1} = x^{-1}$, so this Sylow subgroup is isomorphic to the quaternion group. Its character table is as follows:

Q	1	x	x^2	y	xy
θ_1	1	1	1	1	1
θ_2	1	1	1	-1	-1
θ_3	1	-1	1	1	-1
θ_4	1	-1	1	-1	1
θ_5	2	0	-2	0	0

(The resemblence to the character table of D_8 is uncanny, but the two groups are nonisomorphic.) If we are to have any hope of obtaining an induced representation of G that has nontrivial central character, we must induce a character with nontrivial central character of Q. (Luckily Q and G have the same center $\{\pm I\}$.) There is only one candidate, so we try inducing θ_5 to G. We choose 3 arbitrary right coset representatives $x_1, \cdots, x_3$; then $\sum \dot{\theta}_5(x_i g x_i^{-1})$ is easy to compute, because it *must* vanish unless $g = \pm I$, since $\dot{\theta}_5$ vanishes on the noncentral elements. However if $g = \pm I$ it is central, so

$$\theta_5^G(g) = \sum_{i=1}^{3} \dot{\theta}_5(x_i g x_i^{-1}) = 3\dot{\theta}_5(6),$$

and we have the following characters:

	1 $\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$	1 $\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$	6 $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$	4 $\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$	4 $\begin{pmatrix} 1 & -1 \\ & 1 \end{pmatrix}$	4 $\begin{pmatrix} -1 & -1 \\ & -1 \end{pmatrix}$	4 $\begin{pmatrix} -1 & 1 \\ & -1 \end{pmatrix}$
χ_1	1	1	1	1	1	1	1
χ_2	1	1	1	ρ	ρ^2	ρ	ρ^2
χ_3	1	1	1	ρ^2	ρ	ρ^2	ρ
χ_4	3	3	-1	0	0	0	0
θ_5^G	6	-6	0	0	0	0	0

This representation is, unfortunately, not irreducible, but we will find an irreducible representation inside it. Before we can do that, we take another candidate for H, namely the cyclic subgroup of order 6 generated by

$$u = \begin{pmatrix} -1 & -1 \\ & -1 \end{pmatrix}.$$

We induce the linear character $\psi(u^i) = (-1)^i$, and we obtain

	1 $\begin{pmatrix}1 & \\ & 1\end{pmatrix}$	1 $\begin{pmatrix}-1 & \\ & -1\end{pmatrix}$	6 $\begin{pmatrix} & 1\\ -1 & \end{pmatrix}$	4 $\begin{pmatrix}1 & 1\\ & 1\end{pmatrix}$	4 $\begin{pmatrix}1 & -1\\ & 1\end{pmatrix}$	4 $\begin{pmatrix}-1 & -1\\ & -1\end{pmatrix}$	4 $\begin{pmatrix}-1 & 1\\ & -1\end{pmatrix}$
χ_1	1	1	1	1	1	1	1
χ_2	1	1	1	ρ	ρ^2	ρ	ρ^2
χ_3	1	1	1	ρ^2	ρ	ρ^2	ρ
χ_4	3	3	-1	0	0	0	0
θ_5^G	6	-6	0	0	0	0	0
ψ^G	4	-4	0	1	1	-1	-1

Now let us consider $\theta_5^G - \psi^G$. We will show that this is an irreducible character of G. At the moment, we don't even know that it is a character, since we've taken two characc-ters and subtracted them! How can we show that it is an irreducible character when we don't even know that it is a character? There is a way.

We define a *generalized character* of a finite group G to be a linear combination of characters with integer coefficients (possibly negative). If $\chi_1, \cdots, \chi_h$ are the irreducible characters, then a generalized character is any class function

$$\sum_{i=1}^{h} d_i \chi_i, \qquad d_i \in \mathbb{Z}.$$

This generalized character is a character if and only if the d_i are all nonnegative. The set of generalized characters is closed under both addition *and* subtraction. So although we don't know that $\theta_5^G - \psi^G$ is a character, at least it is a generalized character.

Lemma 2. *Suppose that ϕ is a generalized character. If $\langle \phi, \phi \rangle = 1$ and $\phi(1) > 0$, then θ is an irreducible character.*

Proof. Write $\phi = \sum d_i \chi_i$ as a linear combination of the irreducible characters. Since the χ_i are orthonormal, $\langle \phi, \phi \rangle = \sum d_i^2$. Since the d_i are inegers and $\langle \phi, \phi \rangle = 1$, all bone one of the d_i are zero, and one is ± 1. We have to eliminate the possibility that this nonvanishing d_i is -1. But it is impossible that $\phi = -\chi_i$ since both $\phi(1)$ and $\chi_i(1)$ are positive, so $\phi = \chi_i$. □

In the case at hand, $\theta_5^G - \psi^G$ satisfies the hypothesis of the Lemma, so let us call it χ_5. Then two more degree 2 irreducibles can be obtained using by multiplying this one by one of the linear characters. We thus have the completed character table:

$SL_2(\mathbb{F}_3$	1 $\begin{pmatrix}1 & \\ & 1\end{pmatrix}$	1 $\begin{pmatrix}-1 & \\ & -1\end{pmatrix}$	6 $\begin{pmatrix} & 1\\ -1 & \end{pmatrix}$	4 $\begin{pmatrix}1 & 1\\ & 1\end{pmatrix}$	4 $\begin{pmatrix}1 & -1\\ & 1\end{pmatrix}$	4 $\begin{pmatrix}-1 & -1\\ & -1\end{pmatrix}$	4 $\begin{pmatrix}-1 & 1\\ & -1\end{pmatrix}$
χ_1	1	1	1	1	1	1	1
χ_2	1	1	1	ρ	ρ^2	ρ	ρ^2
χ_3	1	1	1	ρ^2	ρ	ρ^2	ρ
χ_4	3	3	-1	0	0	0	0
χ_5	2	-2	0	-1	-1	1	1
χ_6	2	-2	0	$-\rho$	$-\rho^2$	ρ	ρ^2
χ_7	2	-2	0	ρ	ρ^2	$-\rho$	$-\rho^2$