Math 88Q: Conjugation Continued

Here are some questions about conjugation and related topics, harder than the last batch. I will collect these Tuesday, April 1.

1. Let G and H be groups. Let $f: G \longrightarrow H$ be a homomorphism. Show that if f is bijective, then the inverse map $f^{-1}: H \longrightarrow G$ is also a homomorphism. In this case f is called an *isomorphism* and we say that G and H are *isomorphic*. If $G = H$ then an isomorphism $f: G \longrightarrow G$ is called an *automorphism*.

2. Show that the property of being isomorphic is an equivalence relation.

3. Let $g \in G$ and let $f: G \longrightarrow G$ be the map $f(x) = gxg^{-1}$. Show that f is a homomorphism, in fact an isomorphism. Thus we say that *conjugation by g is an automorphism* of G.

4. Let G and H be groups and let $f: G \longrightarrow H$ be a homomorphism. Define $K \subseteq G$ and $M \subseteq H$ as follows.

$$K = \{g \in G \,|\, f(g) = 1\}, \text{ called the } \textit{kernel of } f,$$

$$M = \{f(g) \,|\, g \in G\}, \text{ called the } \textit{image of } f, \text{ sometimes denoted } M = f(G).$$

Show that K and M are subgroups of G and H, respectively. Show that f is injective if and only if $K = \{1\}$ and surjective if and only if $M = H$.

5. Let G and H be groups and let $f: G \longrightarrow H$ be a homomorphism. Let K be the kernel of f. Show that if $g \in G$ and $x \in K$ then $gxg^{-1} \in K$. We express this fact by writing $gKg^{-1} \subseteq K$. By definition if K is a subgroup such that $gKg^{-1} \subseteq K$ for all g, then K is called a *normal* subgroup. Thus a subgroup is normal if and only if it is *closed under conjugation*.